Chapter 1

Combinatorics

1.1 Factorials

Definition 1.1.1. A factorial is the product of all positive integers less than or equal to a given positive integer. In other words $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$.

1.2 Permutations

Definition 1.2.1. A permutation is a possible arrangement of objects in a set where the order of objects matter.

Theorem 1.2.2 (Number of Orderings of Objects)
The number of ways of arranging n objects in a line is $n!$

The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct is $(n-1)!$

The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct and reflections of the same arrangement aren't considered distinct is $\frac{(n-1)!}{2}$

Remark 1.2.3
The reason that this is true is because we can simply fix 1 person to be at the top and there are $(n-1)!$ ways to arrange the other people. This accounts for rotations since rotating an arrangement will result in someone else on top. We divide by 2 for reflections because of symmetry on both the left and right sides of the person chosen to be at the

top.

Theorem 1.2.4 (Permutations Formula)
The number of ways to arrange k objects out of n total objects is

$$^nP_k = \frac{n!}{(n-k)!}$$

1.3 Word Rearrangements

Theorem 1.3.1 (Word Rearrangements)
The number of ways to order a word is

$$\frac{n!}{d_1! \times d_2! \times d_3! \times \dots}$$

where n is the number of letters and $d_1, d_2, d_3, \dots$ are the number of times each of the letters that occur more than 1 time appear in the word.

Remark 1.3.2
This is not only true for words! The number of ways of arranging objects or anything else is also the same.

1.4 Combinations

Definition 1.4.1. A combination is a possible arrangement in a collection of items where the order of the selection does not matter.

Remark 1.4.2
Usually, the words ”permute”, ”order does matter”, etc. imply a permutation while the words ”choose”, ”select”, ”order doesn’t matter”, etc. imply a combination.

Theorem 1.4.3 (Combinations Formula)
The number of ways to choose k objects out of a total of n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

This is typically spoken as "n choose k"

Remark 1.4.4
Notice that

$$\binom{n}{k} = \binom{n}{n-k}$$

This is true because we can see choosing k objects on the left hand side is equivalent to the $n-k$ objects that will not be selected on the right hand side.

1.5 Subsets

Theorem 1.5.1
The number of subsets of a set of size n is 2^n.

Remark 1.5.2
We have 2 choices for each element in the set: whether to include or not include the element in our subset, and since there are n elements in the set, the total number of subsets is $2 \times 2 \times 2 \times \cdots \times 2$ (n times).

Note: This means that one of our subsets is the empty subset, where we decide to not include all of the elements. Remember to check the problem wording whether we should count the empty subset as valid or not.

1.6 Probability

Definition 1.6.1. Probability is the chance something occurs.

Theorem 1.6.2 (Probability)

$$\text{probability} = \frac{\text{Total number of Desired Outcomes}}{\text{Total Outcomes}}$$

1.7 Distinguishability

Remark 1.7.1
In probability problems, whether you decide to multiply for order or not is up to you as both will give the same probability.

Remark 1.7.2
If you decide not to multiply for order when doing probability problems, make sure the number of orderings is consistent across all cases. In the first problem, every pair had 2 orderings. The same would be true for groups of 5 for example as there are always 5! ways to order the elements in the group. However, in the 2nd problem, the two cases had different number of orderings so this doesn't work.

1.8 Probability of Independent Events

Remark 1.8.1

Whenever we need to find the probabilities of 2 independent events happening (the results of the events don't depend on each other), we can simply find the probability of each event individually and multiply them together.

1.9 Probability of Dependent Events

Remark 1.9.1

When the events are dependent, the probability of both events happening (say A and B) is the probability of event A happening times the probability that event B happens given that event A already happened.

1.10 Casework

Many counting or probability problems can be solved by dividing a problem into several cases and calculating arrangements and probabilities for each case before summing them together.

Concept 1.10.1

When doing casework, always try to minimize the number of cases you have to deal with.

1.11 Complementary Counting

Complementary counting is the problem solving technique of counting the opposite of what we want and subtracting that from the total number of cases. The keyword “at least” indicates that complementary counting may be helpful.

Remark 1.11.1
Often times, we have to use casework in conjuction with other techniques like complementary counting.

1.12 Principle of Inclusion and Exclusion

Definition 1.12.1. The principle of inclusion and exclusion (PIE) is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

Definition 1.12.2 (Union Symbol)**.** $|A \cup B|$ is the union of elements in both A and B (duplicates are only written once)

Definition 1.12.3 (Intersection Symbol)**.** $|A \cap B|$ is the intersection of elements in both A and B (only those elements which are in both sets)

Theorem 1.12.4 (Principle of Inclusion Exclusion for 2 Sets)
Given two sets, $|A_1|$ and $|A_2|$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Basically, we count the number of possibilities in 2 ”things” and subtract the duplicates.

Theorem 1.12.5 (Principle of Inclusion Exclusion for 3 Sets)
Given three sets, $|A_1|$, $|A_2|$, $|A_3|$,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

In this formula, we count the number of possibilities in 3 ”things”, subtract the possibilities that are duplicates in all 3 pairs of sets, and add back the number of duplicates in all 3 sets.

Theorem 1.12.6 (Principle of Inclusion Exclusion Generalized)
Stated more formally, if $(A_i)_{1\leq i\leq n}$ are finite sets, then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

.

1.13 Stars and Bars

Theorem 1.13.1 (Stars and Bars)
The number of ways to place n identical objects into k distinguishable bins is

$$\binom{n+k-1}{n}$$

Remark 1.13.2
Stars and Bars is very useful, and can often be adapted based on situations. For example, if each bin has to have at least 1 object in it we assign 1 object to each bin to start off, and apply our formula to the remaining $n-k$ objects and k distinguishable bins.

Concept 1.13.3 (Stars and Bars With Constraints)
Stars and bars is extremely useful, and can often be adapted based on situations. For example, if each bin has to have at least 1 object in it we assign each bin 1 object to start off with and apply our formula with $n-k$ objects and k distinguishable bins.

1.14 Combinatorial Identities

Theorem 1.14.1
$\binom{n}{k} = \binom{n}{n-k}$

Theorem 1.14.2 (Binomial Identity)
The binomial identity states that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Theorem 1.14.3 (Choosing Odd Even Identity)
This identity states

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$$

1.15 Vandermonde's Identity

Theorem 1.15.1 (Vandermonde's Identity)
Vandermonde's Identity states that

$$\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0} = \binom{m+n}{r}$$

.

Theorem 1.15.2 (Special Case of Vandermonde's Identity)

$$\sum_{i=0}^{k} \binom{k}{i}^2 = \binom{2k}{k}$$

1.16 Pascal's and Hockey Stick Identity

Concept 1.16.1 (Pascal's Triangle)

Pascal's Triangle is a triangular array of binomial coefficients and contains numerous patterns that can be used to make complex calculations much easier.

$$\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&&\\
&&&&&&&&&1&&1&&&&&&&&&\\
&&&&&&&&1&&2&&1&&&&&&&&\\
&&&&&&&1&&3&&3&&1&&&&&&&\\
&&&&&&1&&4&&6&&4&&1&&&&&&\\
&&&&&1&&5&&10&&10&&5&&1&&&&&\\
&&&&1&&6&&15&&20&&15&&6&&1&&&&\\
&&&1&&7&&21&&35&&35&&21&&7&&1&&&\\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&&\\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1&\\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1
\end{array}$$

Pascal's triangle is pictured here. It can be represented in terms of combinations, which is depicted in the image below.

$$\begin{array}{ccccccccccc}
&&&&&\binom{0}{0}&&&&&\\
&&&&\binom{1}{0}&&\binom{1}{1}&&&&\\
&&&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&&&\\
&&\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}&&\\
&\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}&\\
\binom{5}{0}&&\binom{5}{1}&&\binom{5}{2}&&\binom{5}{3}&&\binom{5}{4}&&\binom{5}{5}
\end{array}$$

Theorem 1.16.2 (Pascal's Identity)

Pascal's identity states that

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Theorem 1.16.3 (Hockey Stick Identity)

Hockey Stick Identity states

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Theorem 1.16.4 (Hockey Stick Identity Generalization)

$$\binom{j}{k} + \binom{j+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} - \binom{j}{k+1}$$

1.17 Rectangle Counting

Theorem 1.17.1

The general formula for the number of rectangles of all sizes in a rectangular grid of size $m \times n$ is

$$\binom{m+1}{2} \times \binom{n+1}{2}$$

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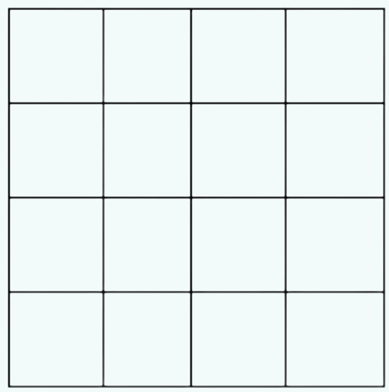


Remark 1.17.2

Each combination of two horizontal lines and two vertical lines creates a unique rectangle. We have

$$\binom{m+1}{2}$$

ways to choose two horizontal lines and

$$\binom{n+1}{2}$$

ways to choose two vertical lines.

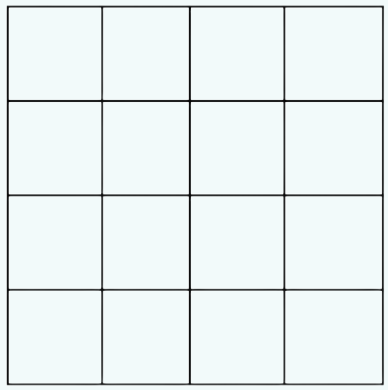
1.18 Square Counting

Theorem 1.18.1

The number of squares in a $n \times n$ grid of squares is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

.



Remark 1.18.2

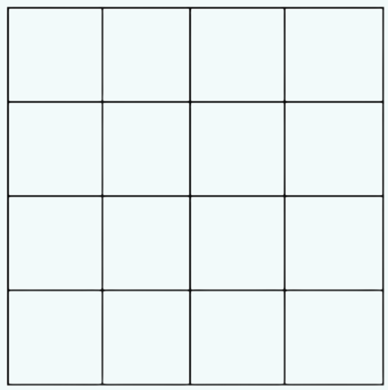
This comes from the fact that there are n^2 1x1 squares, $(n-1)^2$ 2x2 squares, ..., 2^2 (n - 1)x(n - 1) squares, and 1^2 nxn square.

1.19 Path Counting

Theorem 1.19.1

The number of ways to get from (0,0) to a point (m,n) moving only up and right is

$$\binom{m+n}{m}$$



Remark 1.19.2

Imagine any arrangement string of m R's and n U's. Each string would correspond to a unique path, so to count this, we can use the word rearrangement formula.

1.20 Geometric Probability

Definition 1.20.1. Geometric probability is a way to calculate probability by measuring the number of outcomes geometrically, in terms of length, area, or volume. The key to solving geometric probability is

1. Try a few examples for the different cases, make sure to always mark the extreme cases
2. Try to figure out the region the shape maps out
3. Use geometry to find the area of this region

Remark 1.20.2
Geometric probability can be useful when the number of possible outcomes is infinite.

1.21 Expected Value

Definition 1.21.1. Expected value is the weighted average of outcomes.

Theorem 1.21.2 (Expected Value)
The expected value of some event X is

$$\sum x_i \cdot P(x_i)$$

where x_i are the possible values of X and $P(x_i)$ is the probability they occur.
Basically the expected value is just the weighted sum of probabilities of events

Remark 1.21.3
Often times, in finding the expected value, we can just look for symmetry instead of summing each individual probability times number. For example, to calculate the expected value of a dice roll rather than evaluating

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

we can see that since all rolls from 1 to 6 are equally likely, the expected value is just the average roll which is just the average of the 2 middle terms which is 3.5. (See the arithmetic sequences section)

Theorem 1.21.4 (Linearity of Expectation)
For independent or dependent events,

$$E(x_1 + x_2 + \cdots + x_n) = Ex_1 + Ex_2 + \cdots + Ex_n$$

Basically, what this means is that the total expected value of n events is just the sum of the expected values of each individual event.

Remark 1.21.5
This theorem is powerful as it allows us to find the expected value of the individual events rather than of the whole thing at once.

1.22 Recursion

Concept 1.22.1 (Recursion)
Recursion is the process solving the problem for small values and writing a recurrence equation to iteratively calculate the values for larger values.

Steps to Solve Recursion Problems:

1. Base Cases: Manually find the values for small values of n
2. Recursion Equation: Look at the different cases for any general value of n (ex. whether the last digit is 0 or 1)
   - If you are stuck, you can try a few small cases and look for a pattern
3. Iteratively calculate higher values of n until you reach your answer

Remark 1.22.2
Note that you can get the answer to many recursion problems by using engineering induction (see the meta-solving section).

1.23 Probability States

Concept 1.23.1 (States)
We use states in problems when we are trying to find the probability of ”a win” from different positions or turns

When encountering states problems we use the following steps

1. Assign variables to the probabilities of winning from the different positions
2. Write your equations for the probability of winning from each of these positions in terms of the other states
3. Solve your system of equations

Concept 1.23.2 (Symmetry in States)
Always be on the lookout for symmetry in states problems (positions where you have an equal probability of winning from) to help simplify your equations.

Remark 1.23.3
Often times in state problems when you have a lot of states, you may have to write a state recursion equation.

Chapter 2

Algebraic Manipulations

2.1 Algebraic Manipulations

Theorem 2.1.1 (Exponent Rules)

$$x^{-a} = \frac{1}{x^a}$$
$$x^a \times x^b = x^{a+b}$$
$$x^a \div x^b = x^{a-b}$$
$$(x^a)^b = x^{ab}$$

2.2 Binomial Expansions

Theorem 2.2.1 (Binomial Square Expansions)

$$(x+y)^2 = x^2 + 2xy + y^2 = (x-y)^2 + 4xy$$

$$(x-y)^2 = x^2 - 2xy + y^2 = (x+y)^2 - 4xy$$

$$(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$$

$$(x+y)^2 - (x-y)^2 = 4xy$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz)$$

Theorem 2.2.2 (Binomial Cube Expansions)

$$(x+y)^3 = x^3 + 3xy(x+y) + y^3$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3xy(x-y) - y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-xz-yz)$$

Theorem 2.2.3
If
$$x + \frac{1}{x} = a$$
then
$$x^2 + \frac{1}{x^2} = a^2 - 2$$
$$x^3 + \frac{1}{x^3} = a^3 - 3a$$
$$x^4 + \frac{1}{x^4} = (a^2 - 2)^2 - 2$$

2.3 Quadratic Factorizations

Theorem 2.3.1 (Difference of Squares)
$$x^2 - y^2 = (x - y)(x + y)$$

2.4 Cubic Factorizations

Theorem 2.4.1 (Difference of Cubes)

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Theorem 2.4.2 (Sum of Cubes)

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

2.5 Higher Power Factorizations

Theorem 2.5.1 (nth power Factorizations)
Sum of odd powers

$$x^{2n+1} + y^{2n+1} = (x + y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots - xy^{2n-1} + y^{2n})$$

Note: The signs in the second term alternate between positive and negative

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$$

Note: The signs in second term are all positive

Theorem 2.5.2 (Simon's Favorite Factoring Trick)

$$xy + kx + jy + jk = (x + j)(y + k)$$

Remark 2.5.3
You can generally apply this factorization when you have xy, x, and y terms. After applying the factorization, you can then find all possible values for each of your terms in your factorization (remember negatives!).

2.6 Algebraic Equations

Concept 2.6.1 (Algebraic Equations Techniques)

1. Substitution
2. Elimination
3. Adding/Subtracting/Multiplying Equations
4. Group terms together
5. Find common denominator
6. Don't always have to find every variable
7. Use symmetry between variables
8. Make smart substitutions that can simplify your expression
9. Look for common factorizations

2.7 Polynomials

Theorem 2.7.1 (Quadratic Formula)
The solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Concept 2.7.2 (Discriminant)
In the quadratic formula,

$$b^2 - 4ac$$

(the part inside the square root) is the discriminant of the quadratic.

1. If the discriminant $b^2 - 4ac$ is 0, then the quadratic has a double or repeated root
2. If the discriminant $b^2 - 4ac$ is positive, the quadratic has 2 different real roots

3. If the discriminant $b^2 - 4ac$ is negative, the quadratic has no real roots

Theorem 2.7.3 (Vieta's Formula For Quadratics)
In a quadratic equation
$$ax^2 + bx + c = 0$$
the sum of its roots is
$$\frac{-b}{a}$$
and the product of its roots is
$$\frac{c}{a}$$

Theorem 2.7.4 (Vieta's Formula For Higher Degree Polynomials)
In a polynomial
$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$$
with roots
$$r_1, r_2, r_3, ...r_n$$
the following holds:
$$r_1 + r_2 + r_3 + ... + r_n \text{ (the sum of all terms)} = -\frac{a_{n-1}}{a_n}$$

$$r_1r_2 + r_1r_3 + .. + r_{n-1}r_n \text{ (the sum of all products of 2 terms)} = \frac{a_{n-2}}{a_n}$$

$$r_1r_2r_3 + r_1r_2r_4 + ... + r_{n-2}r_{n-1}r_n \text{ (the sum of all products of 3 terms)} = -\frac{a_{n-3}}{a_n}$$

$$\vdots$$

$$r_1r_2r_3 \dots r_n \text{ (the sum of all products of n terms)} = (-1)^n \frac{a_0}{a_n}$$

Note that the negative and positive signs alternate. When summing the products for odd number of terms, we will have a negative sign otherwise we will have a positive sign.

2.8 Polynomial Root Representation

Theorem 2.8.1 (Representation of Polynomial in terms of roots)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$$
it can be expressed in the form
$$a_n(x - r_1)(x - r_2)(x - r_3)\dots(x - r_n)$$
where $r_1, r_2, r_3, \dots, r_n$ are the n roots of the polynomial.

Corollary 2.8.2 (Representation of Monic Polynomial in terms of roots)
In a polynomial
$$P(x) = x^n + a_{n-1}x^{n-1} + \dots a_1 x^1 + a_0$$
(where the leading coeffecient is 1), it can be expressed in the form
$$(x - r_1)(x - r_2)(x - r_3)\dots(x - r_n)$$
where $r_1, r_2, r_3, \dots, r_n$ are the n roots of the polynomial.

Concept 2.8.3
When a problem asks you to find an expression like $(k - r)(k - s)(k - t)$ where r, s, and t are roots of the polynomial for a monic polynomial, it would just be equal to $P(k)$ by the above definition. The same will work for non monic polynomials except it would be $\frac{P(k)}{a_n}$ where a_n is the coefficient of the x^n term in the polynomial.

Theorem 2.8.4 (Fundamental theorem of Algebra)
A polynomial of degree n (the largest term is to the power of n) has n complex roots including multiplicity (for example, a double root would be counted as 2 roots when including multiplicity)

Theorem 2.8.5 (Remainder Theorem)
The remainder when when a polynomial $P(x)$ is divided by $x - r$ is $P(r)$

Corollary 2.8.6 (Factor Theorem)
$x - r$ will divide a polynomial $P(x)$ if $P(r) = 0$
This is a direct consequence of the remainder theorem.

2.9 Symmetric Polynomials

Definition 2.9.1 (Symmetric Polynomials)**.** A polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

is symmetric if

$$a_n = a_0$$
$$a_{n-1} = a_1$$
$$a_{n-2} = a_2$$
$$a_{n-3} = a_3$$
$$\text{etc.}$$

Basically, opposite coefficients are equal.

Concept 2.9.2 (Solving Symmetric Polynomials of Even Degree)
To solve a symmetric polynomial $P(x) = a_0 x^n + a_1 x^{n-1} + ... + a_1 x + a_0$ of even degree,

- Divide by $x^{\frac{n}{2}}$
- Group the x^k and $\frac{1}{x^k}$ terms together
- Make the substitution
$$y = x + \frac{1}{x}$$
and write all the terms in your expression that way
- Solve the reduced polynomial

2.10 Polynomial Manipulations

Concept 2.10.1 (Reciprocal Roots)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$
with roots $r_1, r_2, r_3, \ldots, r_n$,
$$Q(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n$$
will have roots
$$\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n}$$

Essentially, when flipping the coefficients of a polynomial, it will have roots that are reciprocals of the original roots.

Concept 2.10.2 (Roots That Are More or Less)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$
with roots $r_1, r_2, r_3, \ldots, r_n$,
$$Q(x) = a_n(x-k)^n + a_{n-1}(x-k)^{n-1} + ... + a_1(x-k) + a_0$$
will have roots
$$r_1 + k, r_2 + k, r_3 + k, \ldots, r_n + k$$

Remark 2.10.3
Remember, if the roots are k more, than we subtract k from each of the x terms in our polynomial.

Remark 2.10.4
Polynomial manipulations are useful when evaluating complex expressions in terms of roots. For example, in order to evaluate
$$\frac{1}{(r-3)^3} + \frac{1}{(s-3)^3} + \frac{1}{(t-3)^3}$$
of a polynomial with roots r, s, t, rather than expanding it out and bashing with Vieta's

Formulas, we can simplify construct a new polynomial with roots

$$\frac{1}{r-3}, \frac{1}{s-3}, \frac{1}{t-3}$$

. Then, we can use standard Vieta tricks to find $r^3 + s^3 + t^3$. One way is to use the factorization $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$ in the new polynomial.

Theorem 2.10.5 (Newton Sums)
In a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

with roots

$$r_1, r_2, r_3, \ldots, r_n$$

let $S_1 = r_1 + r_2 + \cdots + r_n$

$S_2 = r_1^2 + r_2^2 + \cdots + r_n^2$

$\vdots$

$S_k = r_1^k + r_2^k + \cdots + r_n^k$

$\vdots$

then the following holds true

$a_n S_1 + a_{n-1} = 0$

$a_n S_2 + a_{n-1} S_1 + 2a_{n-2} = 0$

$a_n S_3 + a_{n-1} S_2 + a_{n-2} S_1 + 3a_{n-3} = 0$

$\vdots$

Note that $a_{\text{negative number}} = 0$ as that might show up in your expansion.

Essentially, what this is saying is

1. Start off with a S_k value and multiply by it by the leftmost polynomial coefficient.
2. Then, multiply S_{n-1} by the polynomial's coefficient right after it.
3. Continue doing so and summing the products until either
   - $k = 0$ in which case instead of multiplying S_0 by last the last a term we multiply k
   - a_{n-i} becomes 0 in which case we simply add the last term and stop
4. Set your final sum of terms to be equal to 0

Remark 2.10.6
Note that by each Newton Sum Equation, we can iteratively calculate each P_k rather than having to bash with Vieta's Formulas.

2.11 Arithmetic Sequences

Definition 2.11.1 (Arithmetic Sequence)**.** An arithmetic sequence is a sequence of numbers with the same difference between consecutive terms.

$$1, 4, 7, 10, 13, \ldots, 40$$

is an arithmetic sequence because there is always a difference of 3 between consecutive terms.

Remark 2.11.2
Note that an arithmetic sequence can also have a negative common difference. For example, in the arithmetic sequence

$$40, 37, 34, \ldots, 4, 1$$

the common difference is -3.

Definition 2.11.3 (Arithmetic Sequence Notation)**.** In general, the terms of an arithmetic sequence can be represented as:

$$a_1, a_2, a_3, a_4, \ldots, a_n$$

where

- d is the common difference between consecutive terms
- n is the number of terms

Theorem 2.11.4 (nth term in an Arithmetic Sequence)

$$a_n = a_1 + (n-1)d$$

$$a_n = a_m + (n-m)d$$

Theorem 2.11.5 (Number of terms in an Arithmetic Sequence)

$$n = \frac{a_n - a_1}{d} + 1$$

Essentially,

$$\text{Number of Terms} = \frac{\text{Last Term} - \text{First Term}}{\text{Common Difference}} + 1$$

Theorem 2.11.6 (Average of terms in an Arithmetic Sequence)

$$\text{Average of Terms} = \frac{a_1 + a_n}{2}$$

Theorem 2.11.7 (Sum of all terms in an Arithmetic Sequence)

$$S_n = \frac{a_1 + a_n}{2} \times n$$

Substitute

$$a_n = a_1 + (n-1)d$$

to get

$$S_n = \frac{2a_1 + (n-1)d}{2} \times n$$

2.12 Geometric Sequences

Definition 2.12.1 (Geometric Sequence)**.** A geometric sequence is a sequence of numbers with the same ratio between consecutive terms.

$$1, 2, 4, 8, 16, 32 \ldots, 1024$$

is a geometric sequence because there is always a ratio of 2 between consecutive terms.

Definition 2.12.2 (Geometric Sequence Notation)**.** In general, the terms of a geometric sequence can be represented as:

$$g_1, g_2, g_3, g_4, \ldots, g_n$$

where

- r is the common ratio between consecutive terms
- n is the number of terms

Remark 2.12.3
Note that a geometric sequence can also have a negative common ratio. For example the sequence $1, -2, 4, -8, \ldots, 256, -512, 1024$ has a common ratio of -2.

Theorem 2.12.4 (nth term in a Geometric Sequence)

$$g_n = g_1 \cdot r^{n-1}$$
$$g_n = g_m \cdot r^{(n-m)}$$

Theorem 2.12.5 (Sum of all terms in a finite Geometric Sequence)

$$S_n = g_1 \frac{(r^n - 1)}{r - 1}$$

Theorem 2.12.6 (Sum of all Terms in an Infinite Geometric Sequence)
For $-1 < r < 1$,

$$S_\infty = \frac{g_1}{1 - r}$$

Remark 2.12.7
The reason the formula only works for $|r| < 1$ is because if $|r| \geq 1$ the sum will diverge or essentially be infinite. We can only find the sum of an infinite geometric sequence which is converging as its sum approaches a constant value. Examples:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$$

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

2.13 Special Sequences

Theorem 2.13.1 (Sum of numbers formula)

$$1 + 2 + 3 + \cdots + n = \frac{(n)(n+1)}{2}$$

Theorem 2.13.2 (Sum of odd numbers formula)

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

In simple terms, the

$$\text{Sum of first n odd numbers} = n^2$$

Theorem 2.13.3 (Sum of even numbers formula)

$$2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

To intuitively think about it, just take 2 common from each term

$$2(1 + 2 + 3 + \cdots + n) = 2\frac{(n)(n + 1)}{2} = n(n + 1)$$

Theorem 2.13.4 (Sum of squares formula)

$$1^2 + 2^2 + \cdots + n^2 = \frac{(n)(n + 1)(2n + 1)}{6}$$

Theorem 2.13.5 (Sum of Cubes Formula)

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{(n)(n + 1)}{2}\right)^2$$

2.14 Telescoping

Concept 2.14.1 (Telescoping)
Expand the first few and last few terms, and cancel out any terms you see.

Remark 2.14.2
Generally, whenever you have long expressions that seem to be hard or impossible to compute manually, telescoping is probably a good option.

Concept 2.14.3 (Partial Fraction Decomposition)
Partial fraction decomposition is a telescoping technique in where you split terms into multiple terms in order for terms to cancel. In general, to find the partial decomposition of $\frac{1}{ab}$ for any arbitrary variables a and b, we write the equation

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{ab}$$

and then solve for x and y. For example, the partial fraction decomposition of

$$\frac{1}{n(n+1)}$$

(in this case $a = n$ and $b = n + 1$ is

$$\frac{1}{n}-\frac{1}{n+1}$$

Using partial fraction decomposition, we can telescope and evaluate expressions easily.

2.15 Mean Median Mode

Definition 2.15.1 (Mean/Average)**.**

$$\text{Mean} = \text{average of all terms} = \frac{\text{sum of all terms}}{\text{number of terms}}$$

Definition 2.15.2 (Median)**.** After arranging the numbers in increasing or decreasing order: If number of terms is odd,

$$\text{Median} = \text{middle number}$$

If number of terms is even,

$$\text{Median} = \text{average of middle two numbers}$$

Definition 2.15.3 (Mode)**.**

$$\text{Mode} = \text{Most common term(s)}$$

Remark 2.15.4
There could be multiple modes. If the problem says "unique mode", it means that there is only one mode.

Definition 2.15.5. Geometric Mean of numbers $a_1, a_2, a_3, \ldots, a_n$

$$\sqrt[n]{a_1 \times a_2 \times a_3 \cdots \times a_n}$$

Definition 2.15.6. Harmonic Mean of numbers $a_1, a_2, a_3, \ldots, a_n$

$$= \frac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}}$$
$$= \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$$

Definition 2.15.7. Quadratic Mean of numbers $a_1, a_2, a_3, \ldots, a_n$

$$= \frac{\sqrt{a_1^2+a_2^2+a_3^2+\cdots+a_n^2}}{n}$$

2.16 Work and Rate

Theorem 2.16.1

$$\text{Work} = \text{Rate} \times \text{Time}$$

Equivalently,

$$\text{Rate} = \frac{\text{Work}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Work}}{\text{Rate}}$$

Theorem 2.16.2

The if one person can do something is a amount of time, and someone else can do it in b amount of time, together they can do it in

$$\frac{ab}{a+b}$$

time.

Remark 2.16.3

This not only applies to work. For example, if a problem says 2 faucets take a and b hours to fill a tub, together they can fill a tub in

$$\frac{ab}{a+b}$$

hours.

2.17 Speed Distance Time

Theorem 2.17.1

$$\text{Distance} = \text{Speed} \times \text{Time}$$

Equivalently,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

Theorem 2.17.2

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

Remark 2.17.3
A common mistake is to assume that average speed is the averages of all speeds (especially when the distance you are traveling at each of those speeds are the same). Remember, that's not true unless you are traveling at those speeds for the same amount of time!

Chapter 3

Number Theory

3.1 Primes

Definition 3.1.1 (Primes)**.** Primes are numbers that have exactly two factors: 1 and the number itself. Ex. 2, 3, 5, 7, 11, 13, 17, 19, 23, etc. are all primes

Remark 3.1.2

In order to check whether a number n is prime, we need to check all the primes that are less than or equal to

$$\sqrt{n}$$

Concept 3.1.3 (Prime Factorization)

Prime factorization is a way to express each number as a product of primes.

Example: prime factorization of 60 is $2^2 \times 3 \times 5$

Theorem 3.1.4 (Number of Factors of a Number)

If the prime factorization of the number is expressed as:

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the number of factors of this number is

$$(e_1 + 1)(e_2 + 1) \ldots (e_k + 1)$$

Remark 3.1.5
Basically, in order to find the number of factors of a number:

1. Find the prime factorization of the number
2. Add 1 to all of the exponents
3. Multiply them together

3.2 Sum of Factors

Theorem 3.2.1 (Sum of Factors of a Number)
If the prime factorization of the number is expressed as:

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the number of factors of this number is

$$(1+p_1^1+p_1^2+\cdots+p_1^{e_1-1}+p_1^{e_1})(1+p_2^1+p_2^2+\cdots+p_2^{e_2-1}+p_2^{e_2})\ldots(1+p_k^1+p_k^2+\cdots+p_k^{e_k-1}+p_k^{e_k})$$

Concept 3.2.2 (Sum of Factors of a Number)
Essentially, for a prime p in the prime factorization, first find the sum of p^k for all possible exponents k in the prime factorization. Then, we will multiply all such sums for all of the primes to get the sum of all the factors of the number.

3.3 Product of Factors

Theorem 3.3.1 (Product of Factors of a Number)
The product of factors of a number n where it has f factors (this can be calculated using the number of factors formula) is $n^{\frac{f}{2}}$

3.4 Divisibility

Concept 3.4.1 (Divisibility Rules)

2	Last digit is even
3	Sum of digits is divisible by 3
4	Last 2 digits divisible by 4
5	Last digit is 0 or 5
6	Divisible by 2 and 3
7	Take out factors of 7 until you reach a small number that is either divisible or not divisible by 7
8	Last 3 digits are divisible by 8
9	Sum of digits is divisible by 9
10	Last digit is 0
11	Calculate the sum of odd digits (O) and even digits (E). If $\|O - E\|$ is divisible by 11, then the number is also divisible by 11
12	Divisible by 3 and 4
15	Divisible by 3 and 5

3.5 GCD and LCM

Definition 3.5.1 (Greatest Common Divisor)**.** The Greatest Common Divisor (GCD) of two or more non-0 integers is the largest positive integer that divides each of the integers. Note: This is also known as GCF (Greatest Common Factor), and the terms GCF and GCD are often used interchangeably.

Definition 3.5.2 (Least Common Multiple)**.** The Least Common Multiple (LCM) of two or more non-0 integers is the smallest positive integer that is divisible by both the numbers.

Concept 3.5.3
GCD/LCM Greatest common divisor of m and n = $GCD(m, n)$ can be found by taking the lowest prime exponents from the prime factorizations of m and n.

Least common multiple of m and n = $LCM(m, n)$ can be found by taking the highest prime exponents from the prime factorizations of m and n.

Theorem 3.5.4
The product of GCD and LCM of two numbers is equal to the product of the two numbers:

$$GCD(m, n) \cdot LCM(m, n) = m \cdot n$$

Theorem 3.5.5
If two numbers have a common factor c, then

$$\gcd(ac, bc) = c \cdot \gcd(a, b)$$

Theorem 3.5.6 (Euclidean Algorithm)
The Euclidean algorithm states that

$$gcd(x, y) = gcd(x - ky, y)$$

where $x > y$ and k is a positive integer.

Remark 3.5.7
We can apply the Euclidean Algorithm multiple times to easily find the GCD of large numbers since after applying the Euclidean algorithm, we know have 2 smaller numbers which we can apply the Euclidean Algorithm again until we get 2 very small numbers.

For example,

$$\begin{aligned} \gcd(186, 92) &= \gcd((186 - (2 \cdot 92)), 92) \\ &= \gcd(2, 92) \\ &= \gcd(2, (92 - (2 \cdot 46))) \\ &= \gcd(2, 0) \\ &= 2 \end{aligned}$$

Theorem 3.5.8 (Bezout's Identity)
Integer solutions to the equation

$$ax + by = c$$

will only exist if and only if $\gcd(a, b)$ divides c

3.6 Modular Arithmetic

Definition 3.6.1.

$$n \equiv a \pmod{b}$$

means the number 'n' leaves the same remainder as 'a' when divided by b

Theorem 3.6.2
If $a = x \pmod{n}$ and $b \equiv y \pmod{n}$, then

$$ab \equiv xy \pmod{n}$$

Theorem 3.6.3
If $a \equiv x \pmod{n}$, then

$$a^m \equiv x^m \pmod{n}$$

Concept 3.6.4 (Digit Cycles)
To calculate large digit(s) of a number a^b, a strategy that may work is to just look for a pattern by computing the first few values of a^b and then seeing that the pattern will repeat for large values of b.

Theorem 3.6.5 (Euler's Totient Function)

If number n has the prime factorization

$$p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \ldots p_n^{e_n}$$

then

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_n}\right)$$

where $\phi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n.

Steps to find totient of a number

1. Find prime factorization
2. For all primes, calculate and multiply

$$1 - \frac{1}{p_i}$$

3. Multiply this product to the number n to get the totient

Theorem 3.6.6 (Euler's Totient Theorem)

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

if and only if

$$gcd(a, n) = 1$$

Corollary 3.6.7 (Fermat's Little Theorem)

$$a^{p-1} \equiv 1 \pmod{p}$$

if an only if p is a prime and

$$gcd(a, n) = 1$$

Concept 3.6.8 (Modular Inverses)

If a is denoted as the modular inverse of $b \pmod{n}$, then

$$ab \equiv 1 \pmod{n}$$

We also write that

$$a^{-1} \equiv b \pmod{n}$$

since a and b are inverses $\pmod{n}$.

Theorem 3.6.9 (Wilson's Theorem)

$$(p-1)! \equiv p-1 \equiv -1 \pmod{p}$$

Theorem 3.6.10 (Binomial Theorem)

For non-negative n,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$

Theorem 3.6.11 (Chinese Remainder Theorem)

If a positive number x satisfies

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

where all n_i are relatively prime, then x has a unique solution $\pmod{n_1 \cdot n_2 \cdot n_3 \ldots n_k}$

Remark 3.6.12

Be careful! This may not necessarily be true if any n_i share common factors as then congruences might contradict each other.

Concept 3.6.13 (Solving Linear Congruences)
To solve a linear congruence with 2 congruences you can either

- Guess and Check until you reach a value that works and satisfies both mods
- Algebraic Method
  1. Find 2 congruences
  $$n \equiv r_1 \pmod{m_1}$$
  $$n \equiv r_2 \pmod{m_2}$$
  such that m_1 and m_2 are relatively prime
  2. Rewrite them algebraically
  $$n = k(m_1) + r_1$$
  $$n = j(m_2) + r_2$$
  3. Set them equal mod the smaller of m_1 and m_2 (in this case, say $m_1 < m_2$)
  $$k(m_1) + r_1 \equiv r_2 \pmod{m_2} \implies m_1 \equiv (r_2 - r_1) \cdot k^{-1} \pmod{m_2}$$
  4. Guess and check to find the value of
  $$k^{-1} \pmod{m_2}$$
  5. Using the value of what b is $\pmod{d}$, rewrite it algebraically.
  6. Substitute it back into the expression
  $$n = k(m_1) + r_1$$
  7. Convert it back to mods to get the final congruence

Concept 3.6.14
The solution to
$$n \equiv r_1 \pmod{m_1}$$
$$n \equiv r_2 \pmod{m_2}$$
is
$$n \equiv r_1 + m_1(r_2 - r_1) \cdot i$$
where $i \equiv m_1^{-1} \pmod{m_2}$

Remark 3.6.15
To solve a general congruence of more than 2 congruences, just solve them 2 at a time until you are left with just 1 congruence.

3.7 Algebraic Number Theory

Concept 3.7.1 (Algebraic Number Theory Techniques)

1. Make substitutions to convert numbers to variables
2. Group terms
3. Look for factorizations
4. Sometimes use modular arithmetic, divisibility, or other number theory techniques

3.8 Quadratic Factorizations

Theorem 3.8.1 (Exponent Rules)

$$x^{-a} = \frac{1}{x^a}$$
$$x^a \times x^b = x^{a+b}$$
$$x^a \div x^b = x^{a-b}$$
$$(x^a)^b = x^{ab}$$

Theorem 3.8.2 (Difference of Squares)

$$x^2 - y^2 = (x-y)(x+y)$$

3.9 Cubic Factorizations

Theorem 3.9.1 (Difference of Cubes)
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Theorem 3.9.2 (Sum of Cubes)
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

3.10 Simon's Favorite Factoring Trick

Theorem 3.10.1 (Simon's Favorite Factoring Trick)

$$xy + kx + jy + jk = (x + j)(y + k)$$

Remark 3.10.2
You can generally apply this factorization when you have xy, x, and y terms. After applying the factorization, you can then find all possible values for each of your terms in your factorization (remember negatives!).

3.11 Higher Power Factorizations

Theorem 3.11.1 (nth power Factorizations)
Sum of odd powers

$$x^{2n+1} + y^{2n+1} = (x + y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots - xy^{2n-1} + y^{2n})$$

Note: The signs in the second term alternate between positive and negative

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$$

Note: The signs in second term are all positive

3.12 Sophie Germain's Identity

Theorem 3.12.1 (Sophie Germain's Identity)

$$x^4 + 4y^4 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$$

Remark 3.12.2
Be on the lookout for 4th powers to apply Sophie Germain's Identity!

3.13 Diophantine Equations

Definition 3.13.1. A Diophantine equation is a polynomial equation such that the only solutions are integers (i.e. all variables have integer values).

Concept 3.13.2 (Strategies to Solve Diophantine Equations)
Here are some ideas on how to solve diophantine equations

- Take mods of different numbers. This is generally useful when you
  1. Show there are no solutions to a Diophantine equation
  2. Show that there are only a specific type of solution
- Bound the possible values of different terms, generally useful when there are a finite number of solutions to your Diophantine equations
- Factoring, using the various factorizations (see the algebra section on this), can help find all the solutions
- Make substitutions to simplify your Diophantine equation
- Look for conditions on what must be multiples/divisors of your variables and rewrite your Diophantine equation in terms of that (ex. if you know a is multiple of 6, then can make substitution $a = 6k$)

3.14 Bases

Definition 3.14.1 (Bases). A number expressed in base-n is similar to base 10 except instead of regrouping to a new place value every 10, we regroup every n.

Concept 3.14.2

A number in base n with digits a_m, a_{m-1}, ..., a_2, a_1, a_0 can be written as:

$$a_m a_{m-1} \dots a_2 a_1 a_0$$

This number can be evaluated as

$$a_m n^m + a_{m-1} n^{m-1} + \dots + a_2 n^2 + a_1 n^1 + a_0 n^0$$

Concept 3.14.3

We can also have bases with decimals (both repeating and terminal).

A number in base n with digits a_m, a_{m-1}, ..., a_2, a_1, a_0 before the decimal point and the digits a_{-1}, a_{-2}, ..., a_{-q} after the decimal point can be written as:

$$a_m a_{m-1} \dots a_1 a_0 . a_{-1} a_{-2} \dots a_{-q}$$

This number can be evaluated as:

$$a_m n^m + a_{(m-1)} n^{(m-1)} + \dots + a_1 n^1 + a_0 n^0 + a_{-1} n^{-1} + a_{-2} n^{-2} + \dots a_{-q} n^{-q}$$

3.15 Palindromes

Definition 3.15.1. A palindrome is a number that reads the same forward and backward.

3.16 Chicken McNugget Theorem

Theorem 3.16.1 (Chicken McNugget Theorem)
The maximum value that cannot be expressed as the sum of non-negative multiples of a and b is $ab - a - b$ if a and b are relatively prime.

For relatively prime positive integers a, b ,there are exactly

$$\frac{(a-1)(b-1)}{2}$$

positive integers which cannot be expressed in the form $ma + nb$ where m and n are positive integers.

Remark 3.16.2
This theorem is useful in finding solutions to problems like ”the maximum amount of money that can’t be created with 3 cent and 5 cent coins”.

Theorem 3.16.3 (Legendre’s Theorem)

$$\text{number of factors of p in } n! =$$
$$\text{the number of multiples of p} \leq n + \text{the number of multiples of } p^2 \leq n + \dots$$

Chapter 4

Geometry

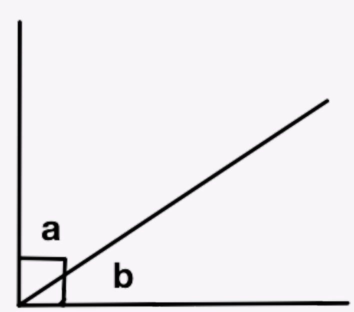
4.1 Angle Chasing

4.2 Basics of Angle Chasing

- Sum of angles in triangle is 180°
- Sum of angles in a line is 180°
- A triangle with 2 equal angles will have their corresponding sides equal and a triangle with 2 sides equal will have their corresponding angles equal. Such a triangle is called an isosceles triangle.

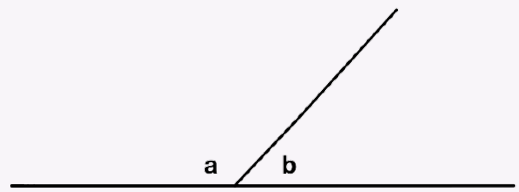
Concept 4.2.1 (Complementary Angle)

Complementary angles are a pair of angles with the sum of 90 degrees



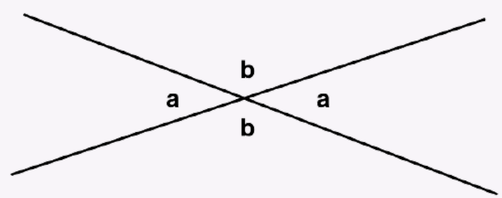
Concept 4.2.2 (Supplementary Angle)

Supplementary angles are a pair of angles with the sum of 180 degrees



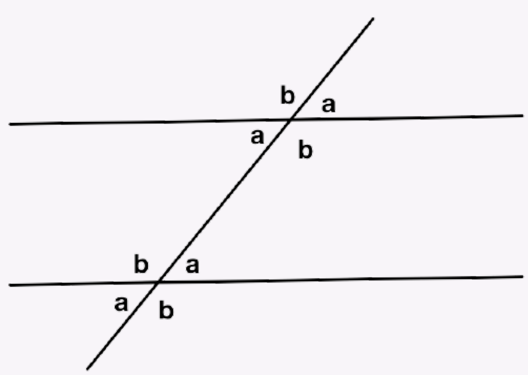
Concept 4.2.3 (Intersecting lines)

When two lines intersect, the vertical angles are equal. Vertical angles are each of the pairs of opposite angles made by two intersecting lines. "Vertical" in this case means they share the same Vertex (corner point), not the usual meaning of up-down.



Concept 4.2.4 (Parallel Lines)

Corresponding angles equal



Theorem 4.2.5

$$\text{Sum of interior angle of a polygon} = (n-2)\cdot 180$$

$$\text{Interior angle of a regular polygon} = \frac{(n-2)}{n}\cdot 180$$

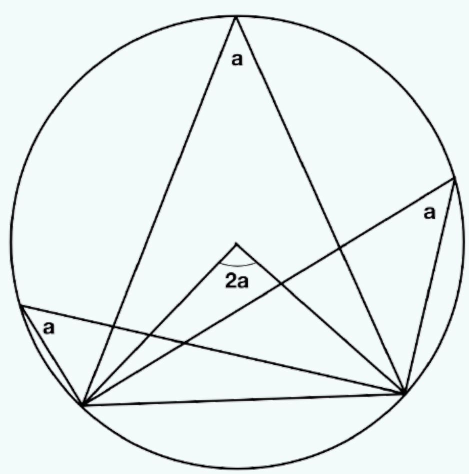
$$\text{Exterior angle of a regular polygon} = \frac{360}{n}$$

Fact 4.2.6. Important Interior Angles

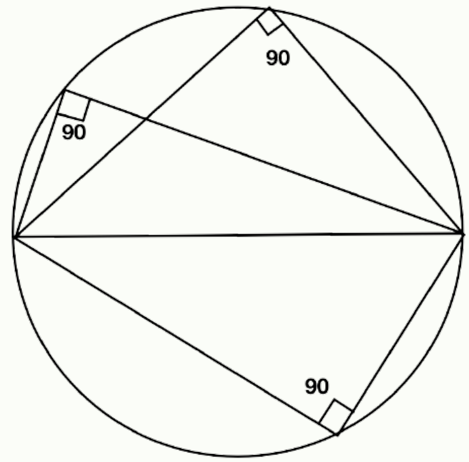
Number of sides in regular polygon	Interior Angle of regular polygon
3	60
4	90
5	108
6	120
8	135
9	140
10	144

Theorem 4.2.7 (Inscribed Arc Theorem)

The angle formed by an arc in the center or the arc angle is double of the angle formed on the edge.



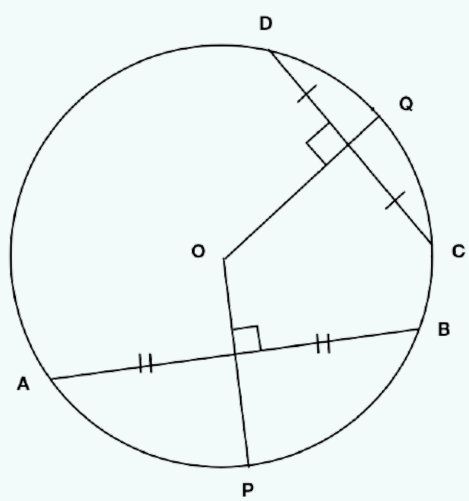
Corollary 4.2.8 (Inscribed Right Triangle)
Inscribed triangle with diameter as one side is always a right triangle.



Definition 4.2.9 (Chord)**.** A Chord is a line segment between any two distinct points on the circle. The diameter of the circle is the longest chord in the circle.

Theorem 4.2.10

The perpendicular bisector of any chord passes through the center. In the figure below, the perpendicular bisectors of AB and CD intersect at the center O.



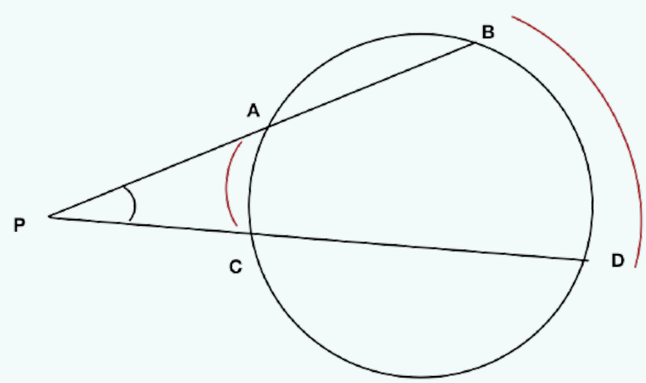
Corollary 4.2.11

- Congruent chords are equidistant from the center of a circle.
- If two chords in a circle are congruent, then their intercepted arcs are congruent.
- If two chords in a circle are congruent, then they determine two central angles that are congruent.

Theorem 4.2.12

The angle marked in the diagram is half of the difference of the 2 red arcs.

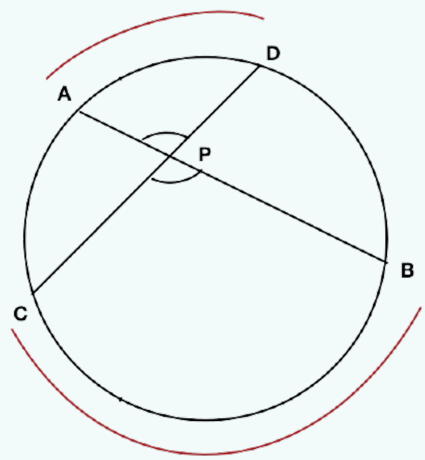
$$\angle APC = \frac{\widehat{\mathrm{BD}} - \widehat{\mathrm{AC}}}{2}$$



Theorem 4.2.13

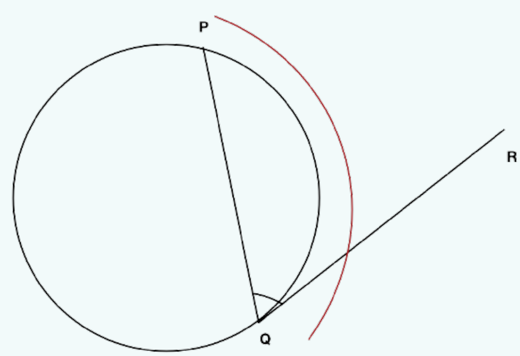
If two chords AB and CD intersect at P, then the $\angle BPC$ and $\angle APD$ are equal to the average of the two arcs.

$$\angle BPC = \angle APD = \frac{\overset{\frown}{\mathrm{BC}} + \overset{\frown}{\mathrm{AD}}}{2}$$



Theorem 4.2.14

If a tangent R intersects the circle at Q, and a chord QP is drawn, then the $\angle RQP$ is equal to half the arc angle



Remark 4.2.15
Circles are really useful for angle chasing so keep an eye out for the inscribed arc theorem

that can be used in many angle chasing problems.

Remark 4.2.16
A useful trick to solving angle chasing problems with regular polygons is to draw a circle around the polygon and use the inscribed arc theorem.

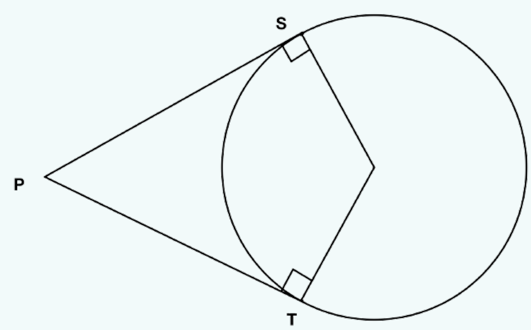
Theorem 4.2.17

Equal chords mark out equal arcs
This basically means that if you have 2 chords of the same length, the sector of the circle they mark out will be equal

Definition 4.2.18 (Tangent)**.** A tangent is any line from a point external to the circle that just touches the circle.

Theorem 4.2.19 (Right Angle Tangency Point)

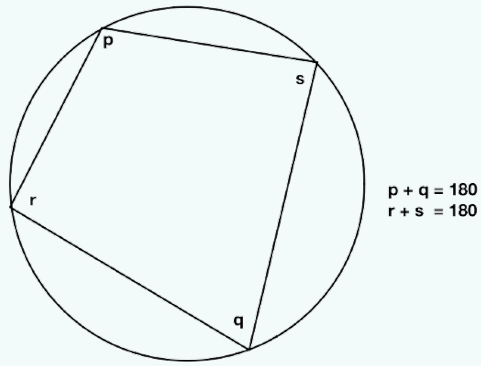
If you connect the center of a circle to the point where the circle and a line are tangent, they will form a right angle.



Remark 4.2.20
This property is very useful in circle problems as it allows us to work with right angles. In addition, another helpful technique is drawing useful radii to various points in your diagram as that opens up new information to work with.

Theorem 4.2.21 (Properties of a Cyclic quadrilateral)

$$\text{Sum of opposite angles} = 180$$

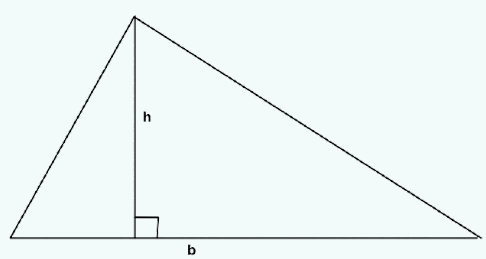


4.3 Area of a Triangle

There are many ways to calculate the area of a triangle. Here are some of the most useful formulas for calculating the area of a triangle:

Theorem 4.3.1 (Using base and height)
A triangle with base b and height h has an area of

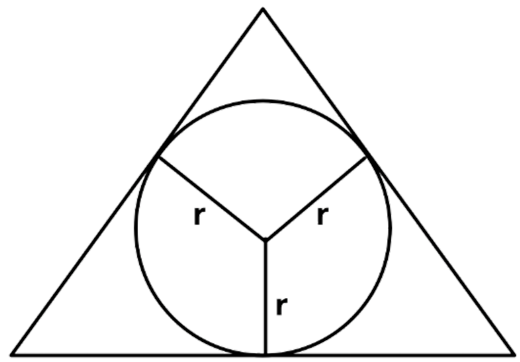
$$\frac{1}{2} \cdot b \cdot h$$



Theorem 4.3.2 (Heron's Formula)
A triangle with sides a, b, c and semiperimeter s has an area of

$$\sqrt{s(s-a)(s-b)(s-c)}$$

Definition 4.3.3 (Incenter). The incenter of a triangle is the intersection of all the angle bisectors. This point is also the center of the incircle, and equidistant from all the three sides.



Definition 4.3.4 (In-radius). The inradius of a triangle is the radius of the inscribed circle in the triangle.

Theorem 4.3.5
Inradius r of a right triangle:

$$r = \frac{1}{2}(a + b - c)$$

where a and b are the legs of the triangle, and c is the hypotenuse.

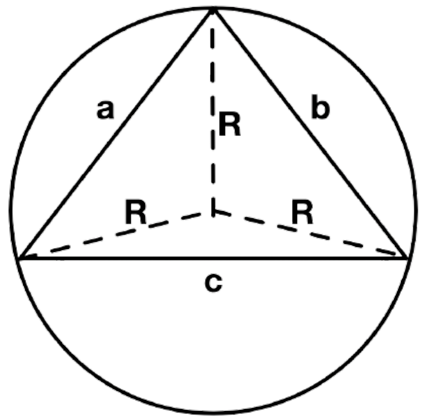
Theorem 4.3.6 (Using inradius)
A triangle with inradius r (the radius of the circle that can be inscribed in a triangle) and semiperimeter s has an area of:

$$Area = \text{inradius} \cdot \text{semiperimeter}$$

$$A = rs$$

Remark 4.3.7
Note that if we know the area of the triangle and it's semi-perimeter, we can apply the inradius formula to find the inradius of the triangle.

Definition 4.3.8 (Circumcenter)**.** The circumcenter of a triangle is the intersection of all 3 perpendicular bisectors (the line that bisects a segment and is perpendicular to it). This point is also the center of the circumcircle and equidistant from all the three vertices.



Definition 4.3.9 (Circumradius)**.** The circum-radius of a triangle is the radius of circle that a triangle is inscribed in.

Theorem 4.3.10 (Using circumradius)

A triangle with circumradius R (the radius of the circle that the triangle can be inscribed in) and sides a, b, c has an area of

$$A = \frac{abc}{4R}$$

Remark 4.3.11

Similar to the inradius problem, if we know all 3 sides of a triangle, we can apply Heron's and easily calculate the circumradius of the triangle.

Theorem 4.3.12 (Shoelace Theorem using coordinates)

Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area (A) of P is

$$A = \frac{1}{2}\left|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)\right|$$

You can also go counterclockwise order, as long as you find the absolute value of the answer.

The Shoelace Theorem gets its name because if one lists the coordinates in a column,

$$\begin{array}{c} (a_1, b_1) \\ (a_2, b_2) \\ \vdots \\ (a_n, b_n) \\ (a_1, b_1) \end{array}$$

and marks the pairs of coordinates to be multiplied,

$$A = \frac{1}{2}\left|\begin{array}{cccccc} a_1 & & b_1 & & a_1 & b_1 \\ a_2 & & b_2 & & a_2 & b_2 \\ \vdots & & \vdots & - & \vdots & \vdots \\ a_n & & b_n & & a_n & b_n \\ a_1 & & b_1 & & a_1 & b_1 \end{array}\right|$$

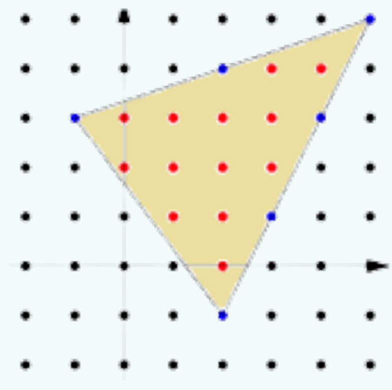
Remark 4.3.13 (Intuitive Way of Thinking about Shoelace Theorem)

Steps to Shoelace Theorem

1. Line up all of your polygon's coordinates in a vertical line
2. Repeat your first coordinate at the bottom of your line
3. Let the sum of products of all rightward diagonally pairs be A
4. Let the sum of products of all leftward diagonally pairs be B
5. Find
$$\frac{1}{2}|A - B|$$

Theorem 4.3.14 (Pick's Theorem)

If a polygon has vertices with integer coordinates (lattice points) then the area of the polygon is $i + \frac{b}{2} - 1$ where i is the number of lattice points inside the polygon and b is the number of lattice points on the boundary of the polygon.

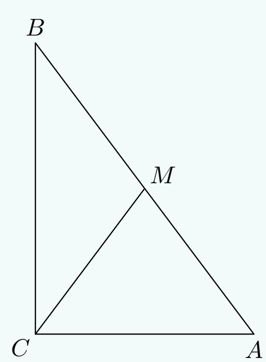


4.4 Triangle Properties

Definition 4.4.1 (Median)**.** A median is a line connecting a point to the midpoint of the opposite side.

Theorem 4.4.2 (Median in a Right Triangle)

In a right triangle ABC, let the median from point C intersect AB at a point M. Then AM = BM = CM.
Basically, in a right triangle AB is the diameter of the circumcircle, and MA, MB, and MC are radii of the circumcircle.



Definition 4.4.3 (Centroid)**.** In a triangle, the intersection of all 3 medians in a triangle is the centroid.

Theorem 4.4.4

The centroid of a triangle is on the median and it is $\frac{2}{3}$ of the way from from one of vertices to the midpoint of the opposite side.

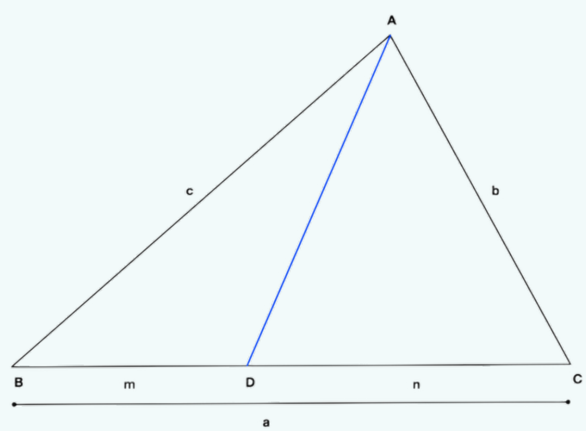
Definition 4.4.5 (Cevian)**.** A cevian is any line from any vertex of a triangle to the opposite side. Medians and angle bisectors are special cases of cevians.

Theorem 4.4.6 (Stewart's Theorem)

Given a triangle $\triangle ABC$ with sides of length a, b, c opposite vertices of A, B, C respectively. If cevian AD is drawn so that $BD = m$, $DC = n$ and $AD = d$, we have that

$$man + dad = bmb + cnc$$

.



Remark 4.4.7

A way to remember this is the saying "A **Man** and his **Dad** put a **Bomb** in the **Sink**

Corollary 4.4.8 (Stewart's For Angle Bisector)

If AD is an angle bisector, then $d^2 + mn = bc$.

Note that this follows from Stewart's Theorem and the Angle Bisector Theorem.

Corollary 4.4.9 (Stewart's Theorem For Medians)
If AD is a median, then $d^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2$

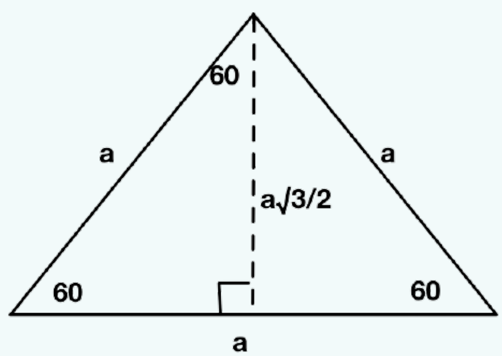
4.5 Equilateral Triangle

Theorem 4.5.1
If the side length of an equilateral triangle is a

$$\text{Height of the triangle} = \frac{\sqrt{3}}{2}a$$

This follows directly from the $30 - 60 - 90$ triangle.

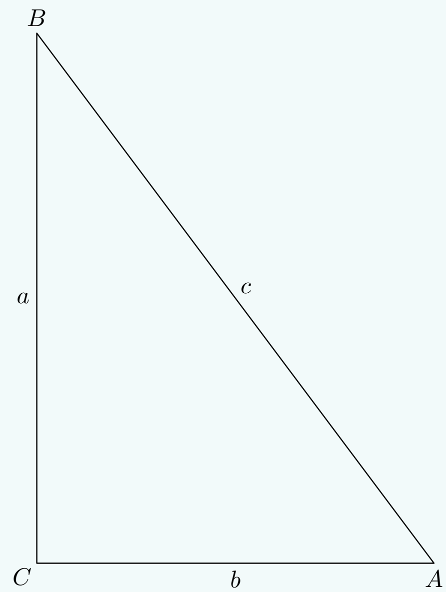
$$\text{Area of the equilateral triangle} = \frac{\sqrt{3}}{4}a^2$$



4.6 Right Triangles

Theorem 4.6.1 (Pythagorean Theorem)
A right triangle with legs a and b and hypotenuse c satisfies the following relation:

$$c^2 = a^2 + b^2$$



Fact 4.6.2. Important Pythagorean Triples
3, 4, 5
5, 12, 13
7, 24, 25
8, 15, 17
9, 40, 41
20, 21, 29

If all numbers in a pythagorean triple are multiplied by a constant, the resulting numbers still form a pythagorean triple.

For example: These are all pythagorean triples:
3, 4, 5
6, 8, 10
9, 12, 15
12, 16, 20
15, 20, 25

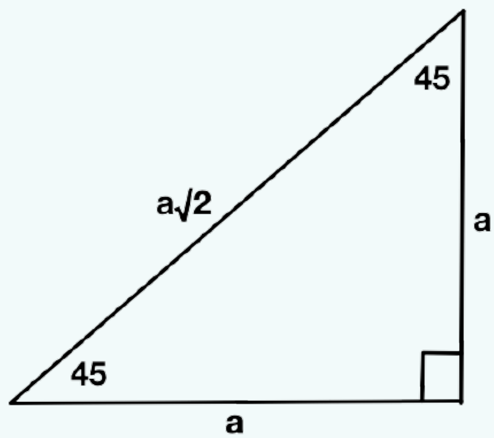
4.6.1 45-45-90 Triangle

Theorem 4.6.3

If the side length of a 45-45-90 triangle is a

$$\text{hypotenuse of the triangle} = \sqrt{2} \times \text{side length} = \sqrt{2}a$$

$$\text{Area of the triangle} = \frac{1}{2} \times \text{side length}^2 = \frac{1}{2}a^2$$



4.6.2 30-60-90 Triangle

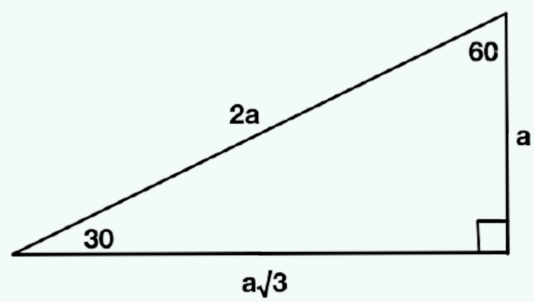
Theorem 4.6.4

If the short leg length of a 30-60-90 triangle is a

$$\text{Long Leg of the triangle} = \sqrt{3} \times \text{short leg} = \sqrt{3}a$$

$$\text{hypotenuse of the triangle} = 2 \times \text{short leg} = 2a$$

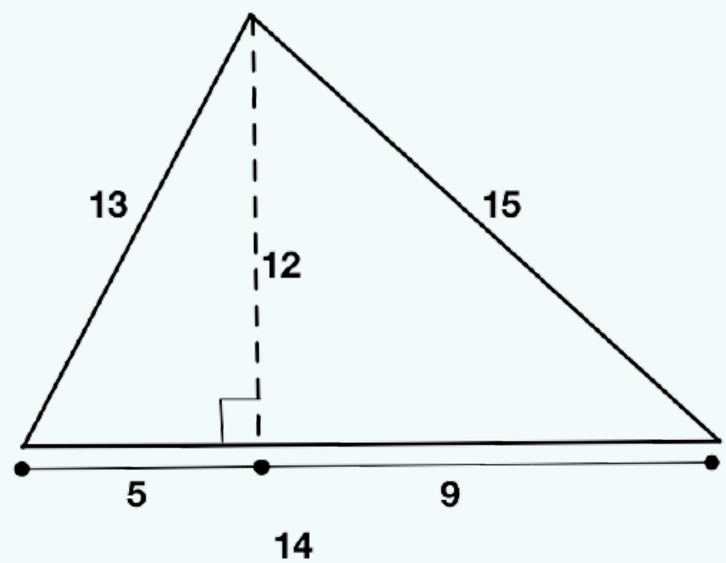
$$\text{Area} = \frac{\sqrt{3}}{2} \times \text{short leg}^2 = \frac{\sqrt{3}}{2}a^2$$



4.6.3 13-14-15 Triangle

Theorem 4.6.5

If the three sides of a triangle are 13, 14, and 15, it can be divided into two right triangles with side lengths:
5, 12, 13 and 9, 12, 15
Area of this triangle $= \frac{1}{2} \times 14 \times 12 = 84$



4.7 Similar Triangles

Theorem 4.7.1

For similar triangles:

1. All the angles of the triangles are same
2. All corresponding sides have same ratio
3. Area ratio is the square of side length ratio

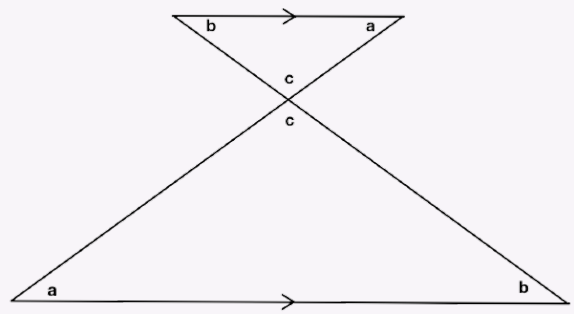
Concept 4.7.2 (Similarity Test)
Two triangles are similar if the three angles in the triangle are the same. In other words,

the triangles are the same shape multiplied by a scale factor. In general, triangles are similar if:

- AA similarity: Two angles of the triangles are same, which basically means that the third angle will be equal)
- SAS similarity (Side Angle Side): Two sides are proportional and the angle between the sides is equal
- SSS similarity (Side Side Side): All three sides are proportional
- HL similarity (Hypotenuse Leg): In a right triangle, hypotenuse and leg are proportional
- LL similarity (LL Leg): In a right triangle, the two legs are proportional

Warning: SSA does not mean triangles are similar

An easy way to detect similar triangles is if bases of triangles are parallel and the sides of the triangles are collinear (see figure below)



Theorem 4.7.3 (Special Properties of right triangles)
In a right triangle ABC where B is the right angle, the following triangles are similar

$$\triangle ABC \sim \triangle ADB \sim \triangle BDC$$

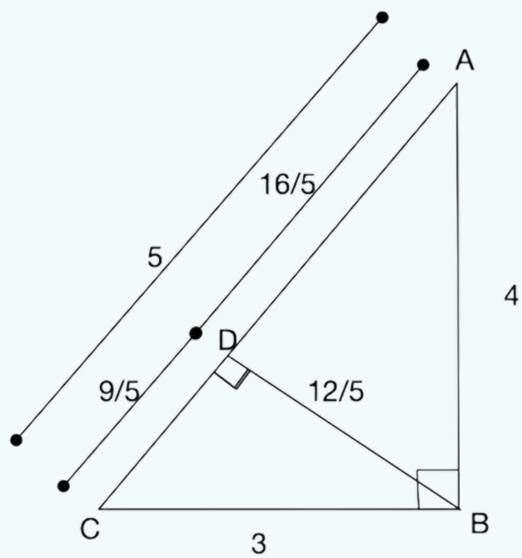
Length of the perpendicular to the hypotenuse (BD) = $\frac{AB \cdot BC}{AC}$

Also note that:

$$\mathrm{AD} \cdot \mathrm{CD} = \mathrm{BD}^2$$

$$\mathrm{AD} \cdot \mathrm{AC} = \mathrm{AB}^2$$

$$\mathrm{CD} \cdot \mathrm{CA} = \mathrm{CB}^2$$

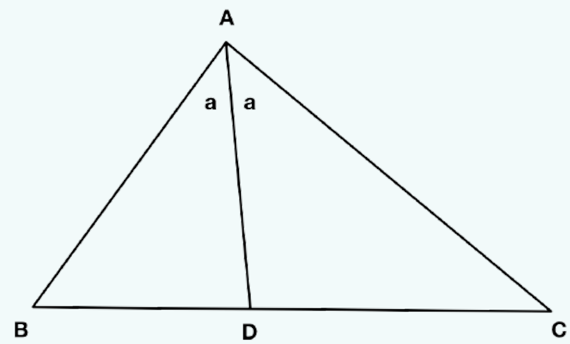


4.8 Angle Bisector Theorem

Theorem 4.8.1 (Angle Bisector Theorem)

If the line AD bisects angle A, then

$$\frac{AB}{BD} = \frac{AC}{CD}$$



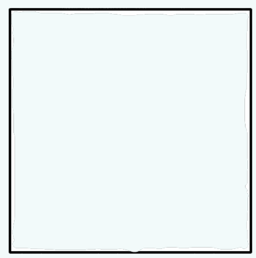
4.9 Square

Theorem 4.9.1 (Area of a Square)
Any square with side length s has an area of

$$s^2$$

and a perimeter of

$$4s$$



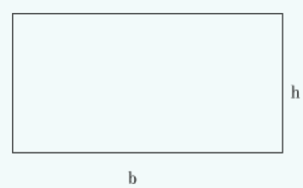
4.10 Rectangle

Theorem 4.10.1 (Area of a Rectangle)
Any rectangle with base b and height h has an area of

$$bh$$

and a perimeter of

$$2b + 2h$$



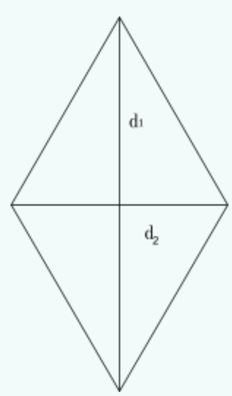
4.11 Rhombus

Theorem 4.11.1 (Area of a Rhombus)
A rhombus with diagonals d_1 and d_2 has an area of

$$\frac{1}{2}d_1d_2$$

and a perimeter of

$$2 \times \sqrt{d_1^2 + d_2^2}$$



Concept 4.11.2 (Rhombus)
1. All sides equal
2. Opposite angles congruent
3. Diagonals are perpendicular bisectors

4.12 Parallelogram

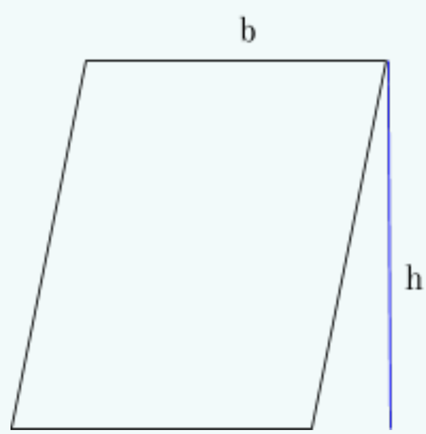
Theorem 4.12.1 (Area of a Parallelogram)
A parallelogram with base b and height h has an area of

$$bh$$

A parallelogram with diagonals d_1 and d_2 has an area of

$$\frac{1}{2}d_1d_2 * sin(\theta)$$

where θ is the central angle of the parallelogram

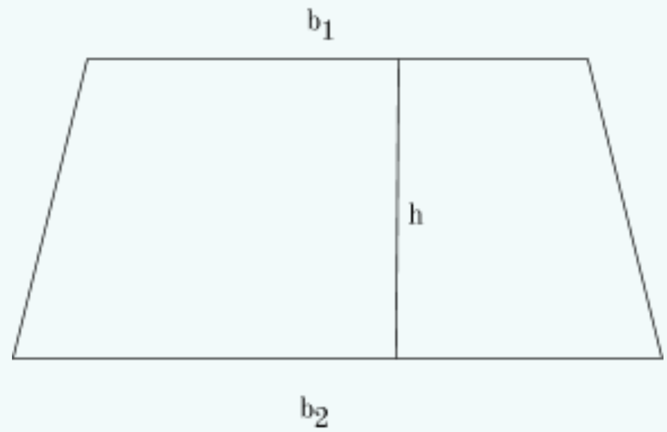


Concept 4.12.2 (Parallelogram Properties)
1. Diagonals bisect each other
2. Opposite sides congruent
3. Opposite angles congruent
4. Base angles supplementary

4.13 Trapezoid

Theorem 4.13.1 (Area of a Trapezoid)
A trapezoid with 2 bases b_1 and b_2 and a height h has an area of

$$\frac{b_1 + b_2}{2} \cdot h$$



4.14 Circle Area

Theorem 4.14.1 (Area and Circumference)
A circle with radius r has

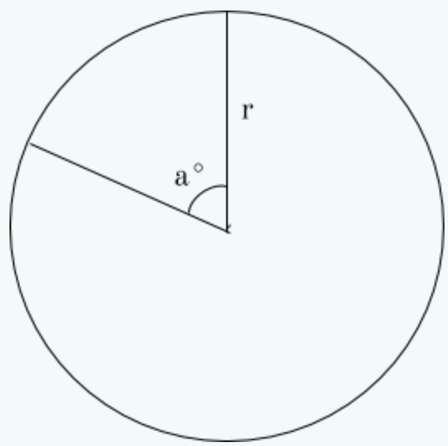
$$\text{Area} = \pi r^2$$

$$\text{Circumference} = 2\pi r$$

Theorem 4.14.2 (Arcs of a circle)
An arc of a circle with radius r and angle $a°$

$$\text{Area of a sector} = \pi r^2 \times \frac{a°}{360} = \pi \times \text{radius}^2 \times \text{fraction of circle in sector}$$

$$\text{Length of the arc} = 2\pi r \times \frac{a°}{360} = 2\pi \times \text{radius} \times \text{fraction of circle in sector}$$



Definition 4.14.3 (Angle of an arc)**.** This is the angle that the arc makes at the center of the circle.

4.15 Circle Length

Concept 4.15.1

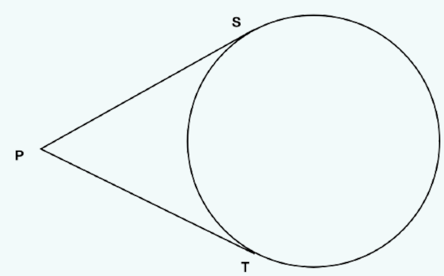
In geometry problems with circles, always try to draw all important radii (i.e. connect center to all points on a circle)

4.16 Power of a Point

Theorem 4.16.1 (Power of Point For 2 Tangents)

From a given point P external to a circle, the two tangents to the circle are equal.

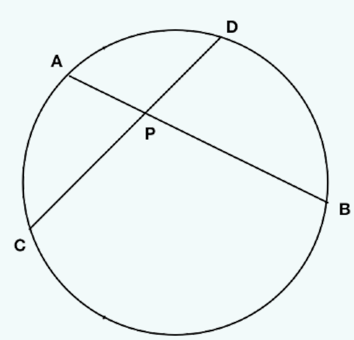
$$PS = PT$$



Theorem 4.16.2 (Power of Point Inside Circle)

If AB and CD are two secants in a circle with Center O, which intersect at a point P, the line segments satisfy the following property:

$$PA \cdot PB = PC \cdot PD = r^2 - OP^2$$

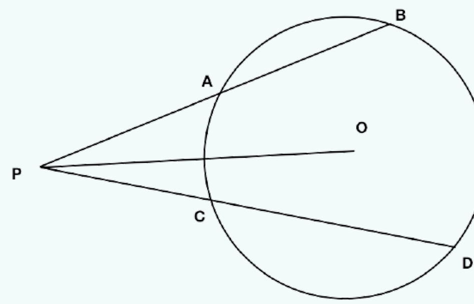


Theorem 4.16.3 (Power of Point Outside Circle)

If AB and CD are two secants in a circle, which intersect at a point P outside the circle with Center O, the line segments satisfy the following property:

$$PA \cdot PB = PC \cdot PD = OP^2 - r^2$$

where r is the radius of the circle



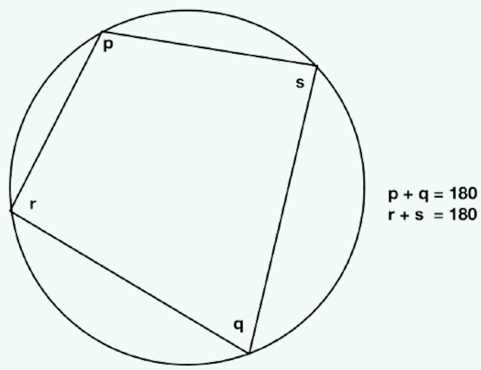
Remark 4.16.4

Power of a point is useful when dealing with circles and chord lengths.

4.17 Cyclic Quadrilaterals

Theorem 4.17.1 (Properties of a Cyclic quadrilateral)

$$\text{Sum of opposite angles} = 180$$



You can draw a circle around quadrilateral and use angle chasing properties for circles like the inscribed angle theorem.

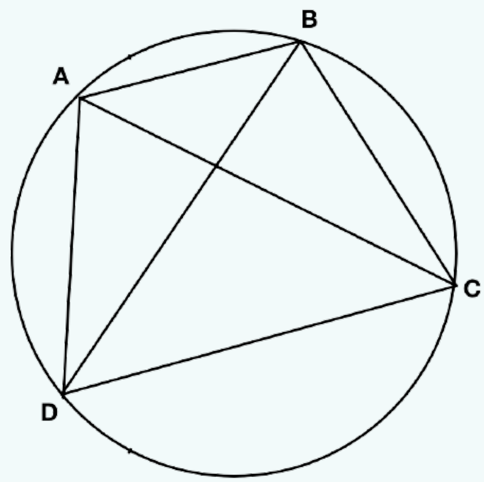
Concept 4.17.2

Sometimes, there may be quadrilaterals not explicitly stated to be in a circle. Then, if the sum of opposite angles is 180° or if the angles satisfy properties from the inscribed angle theorem, then you can identify them to be a cyclic quadrilateral, and you can use any of the properties of cyclic quadrilaterals or even circles because you know the quadrilateral can be inscribed in a circle.

Theorem 4.17.3 (Ptolemy's Theorem)
In a cyclic quadrilateral $ABCD$

$$\text{Product of diagonals} = \text{Sum of the product of both pairs of opposite sides}$$

$$\mathrm{AC} \cdot \mathrm{BD} = \mathrm{AB} \cdot \mathrm{CD} + \mathrm{AD} \cdot \mathrm{BC}$$

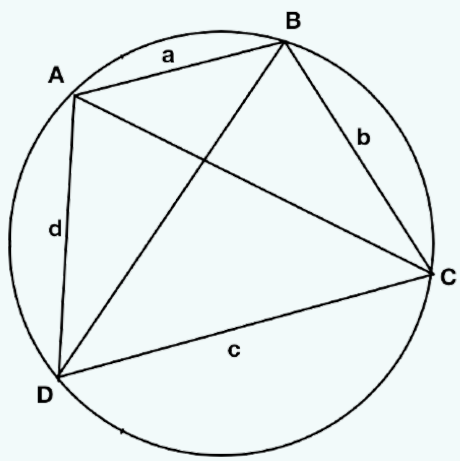


Theorem 4.17.4 (Brahmagupta's Formula)
In a cyclic quadrilateral with side lengths a, b, c, d, the area of the quadrilateral can be found as:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where s is the semiperimeter of the quadrilateral and can be calculated as

$$s = \frac{a+b+c+d}{2}$$



Basically, to find the area of a cyclic quadrilateral

1. Find the perimeter and divide by 2
2. Subtract each of the side lengths from it to get 4 values
3. Multiply your 4 values
4. Take the square root of your product

4.18 Polygons

Theorem 4.18.1

$$\text{Sum of interior angle of a polygon} = (n-2)\cdot 180$$

$$\text{Interior angle of a regular polygon} = \frac{(n-2)}{n}\cdot 180$$

$$\text{Exterior angle of a regular polygon} = \frac{360}{n}$$

Fact 4.18.2. Important Interior Angles

Number of sides in regular polygon	Interior Angle of regular polygon
3	60
4	90
5	108
6	120
8	135
9	140
10	144

4.19 Hexagon

Theorem 4.19.1

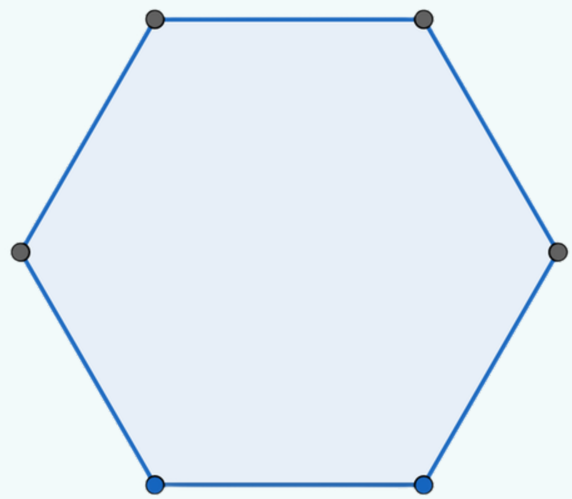
$$\text{Sum of interior angle of a regular hexagon} = (6-2)\cdot 180 = 720$$

$$\text{Interior angle of a regular hexagon} = \frac{(6-2)}{6}\cdot 180 = 120$$

$$\text{Exterior angle of a regular hexagon} = \frac{360}{6} = 60$$

$$\text{Area of a regular hexagon} = 6\cdot\frac{\sqrt{3}}{4}s^2$$

$$\text{Length of the diagonal of a regular hexagon} = 2s$$



4.20 Octagon

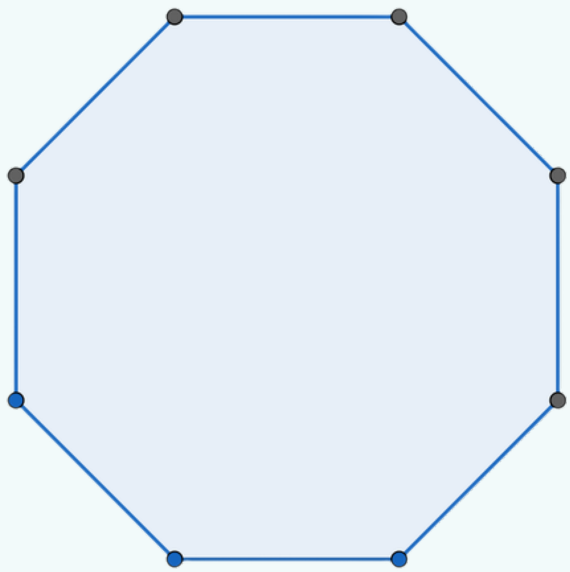
Theorem 4.20.1

$$\text{Sum of interior angle of a regular octagon} = (8-2)\cdot 180 = 1080$$

$$\text{Interior angle of a regular octagon} = \frac{(8-2)}{6}\cdot 180 = 135$$

$$\text{Exterior angle of a regular octagon} = \frac{360}{8} = 45$$

$$\text{Area of a regular octagon} = 2(1+\sqrt{2})s^2$$



Remark 4.20.2
A regular hexagon can be divided into 6 congruent equilateral triangles.

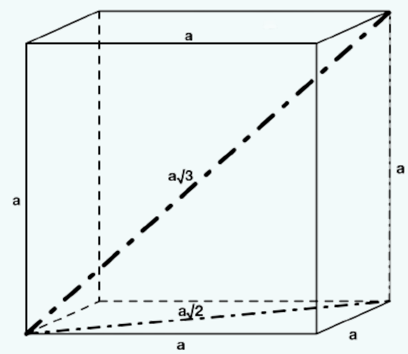
4.21 Pentagons

4.22 Cube

Theorem 4.22.1 (Volume and Surface Area of a cube)

$$\text{Volume of a cube} = (\text{side length})^3 = a^3$$
$$\text{Surface area of a cube} = 6 \times (\text{side length})^2 = 6a^2$$

$$\text{Length of space diagonal of a cube} = \sqrt{3} \times \text{side length} = \sqrt{3}a$$



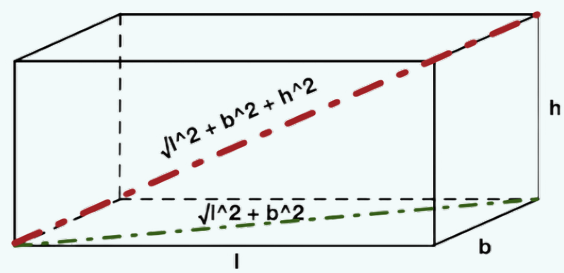
4.23 Rectangular Prism

Theorem 4.23.1 (Volume and Surface Area of a rectangular prism)

$$\text{Volume of a rectangular prism} = l \times b \times h = \text{product of all three dimensions}$$

$$\text{Surface area of a rectangular prism} = 2(lb + bh + lh)$$

$$\text{Length of space diagonal of a rectangular prism} = \sqrt{l^2 + b^2 + h^2}$$

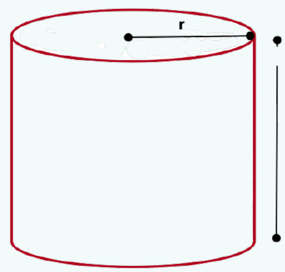


4.24 Cylinder

Theorem 4.24.1 (Volume and Surface Area of a cylinder)

$$\text{Volume of a cylinder} = \pi r^2 h$$

$$\begin{aligned}\text{Surface area of a cylinder} &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r(r+h)\end{aligned}$$



4.25 Cone

Theorem 4.25.1 (Volume and Surface Area of a cone)

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h$$

which basically means

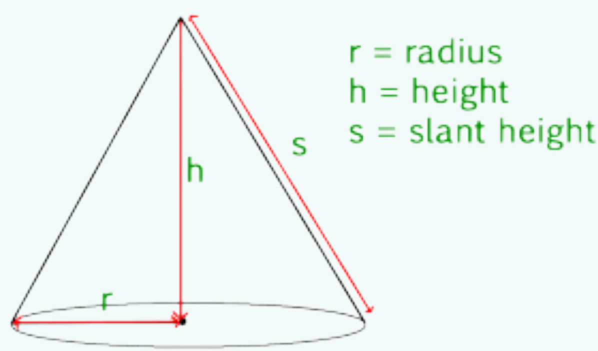
$$\text{Volume of a Cone} = \frac{1}{3}\pi \cdot \text{radius}^2 \cdot \text{height}$$

$$\text{Surface area of a cone} = \pi r^2 + \pi r s = \pi r(r+s)$$

where s is the lateral or slant height

which can also be written as

$$\pi \cdot \text{radius}^2 + \pi \cdot \text{radius} \times \text{slant height}$$



Remark 4.25.2
The slant height s can be calculated by the following formula

$$s = \sqrt{r^2 + h^2}$$

or

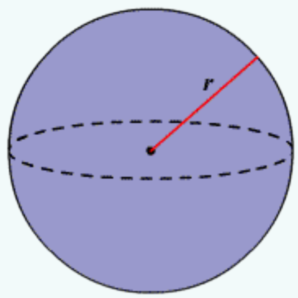
$$\text{slant height} = \sqrt{\text{radius}^2 + \text{height}^2}$$

4.26 Sphere

Theorem 4.26.1 (Volume and Surface Area of a sphere)

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

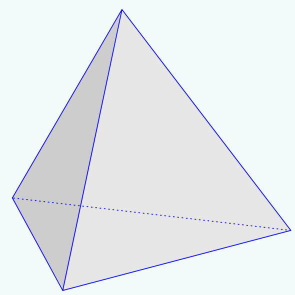
$$\text{Surface area of a sphere} = 4\pi r^2$$



4.27 Tetrahedron

Theorem 4.27.1 (Volume of a tetrahedron)

$$\text{Volume of any tetrahedron} = \frac{1}{3} \cdot \text{base area} \cdot \text{height}$$



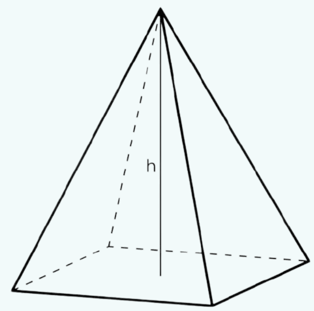
Theorem 4.27.2 (Volume of a regular tetrahedron (all sides equal))

$$\text{Volume of a regular tetrahedron} = \frac{\sqrt{2}}{12} s^3$$

4.28 Pyramid

Theorem 4.28.1 (Volume of a pyramid)

$$\text{Volume of any pyramid} = \frac{1}{3} \cdot \text{base area} \cdot \text{height}$$



Theorem 4.28.2 (Volume of a regular pyramid (all sides equal))
When the pyramid has a square base, and all the sides are equal

$$\text{Volume of a regular pyramid} = \frac{\sqrt{2}}{6}s^3$$

4.29 Volume of Complex Polyhedrons

4.30 Cross Sections

4.31 Area of Complex Shapes

Concept 4.31.1
Tricks to finding the area of complex shapes

- Divide the shape into "nicer" areas which are easier to calculate
- Extend Lines
  - You generally want to extend lines when they form nicer shapes/areas to work

    with, such as triangles

- Break up areas
    - A common way to do so is to drop altitudes as doing so generally allows you to form right triangles

Remark 4.31.2
A common technique is to find the area of shapes and then find the area of a shape in terms of a variable (like altitude, inradius, circumradius, etc.) and then solve for that variable.

4.32 Length of complex shapes

Concept 4.32.1
Finding Length of Complex Shapes

- Having equal angles means equal lengths and vice versa
- Be on the lookout for 90 degree angles, as you can use Pythagorean theorem
- Split the length into multiple components by using some of these techniques
    - Drawing extra lines
    - Dropping Altitudes
- Extending lines to create similar triangles, special triangles, etc. and then subtracting the extra length

4.33 Line in Coordinate Plane

Definition 4.33.1 (Equation of a Line)**.** The equation of a line is $ax + by + c = 0$.

Definition 4.33.2 (Slope-Intercept Form)**.** An equation of a line in slope-intercept form is $y = mx + b$

Theorem 4.33.3 (Slope of a line through 2 points)
The slope of a line passing through 2 points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Theorem 4.33.4 (Slope of a line through angle)
The slope of a line with an angle of θ above the x-axis is $\tan(\theta)$

Theorem 4.33.5 (Distance between 2 points)
The distance between 2 points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Theorem 4.33.6 (Point to line formula)
The distance between a point (x_0, y_0) and a line $ax + by + c = 0$ is

$$\frac{|a \cdot x_0 + b \cdot y_0 + c|}{\sqrt{a^2 + b^2}}$$

.

Remark 4.33.7
Be careful not to get the equation of the line confused with $ax + by = c$

Remark 4.33.8
Note that this distance represents the shortest possible distance which would be length

of the perpendicular line.

Remark 4.33.9
This formula is a bit confusing so an easy way to remember the numerator is that it's just the equation of the line with the values of the point plugged in as the x and y values

4.34 Circle in Coordinate Plane

Theorem 4.34.1 (Equation of a Circle)
A circle with center (a, b) and radius r has equation

$$(x-a)^2 + (y-b)^2 = r^2$$

4.35 Shoelace Theorem to find Area

Theorem 4.35.1 (Shoelace Theorem using coordinates)
Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area (A) of P is

$$A = \frac{1}{2} \left|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)\right|$$

You can also go counterclockwise order, as long as you find the absolute value of the answer.

The Shoelace Theorem gets its name because if one lists the coordinates in a column,

$$\begin{array}{c} (a_1, b_1) \\ (a_2, b_2) \\ \vdots \\ (a_n, b_n) \\ (a_1, b_1) \end{array}$$

and marks the pairs of coordinates to be multiplied,

$$A = \tfrac{1}{2} \left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{array} \quad - \quad \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{array} \right|$$

Remark 4.35.2 (Intuitive Way of Thinking about Shoelace Theorem)
Steps to Shoelace Theorem

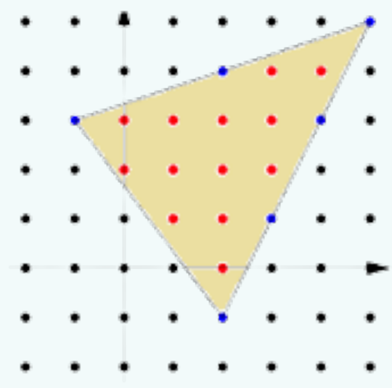
1. Line up all of your polygon's coordinates in a vertical line
2. Repeat your first coordinate at the bottom of your line
3. Let the sum of products of all rightward diagonally pairs be A
4. Let the sum of products of all leftward diagonally pairs be B

5. Find
$$\frac{1}{2}|A-B|$$

4.36 Picks Theorem (Optional)

Theorem 4.36.1 (Pick's Theorem)

If a polygon has vertices with integer coordinates (lattice points) then the area of the polygon is $i+\frac{b}{2}-1$ where i is the number of lattice points inside the polygon and b is the number of lattice points on the boundary of the polygon.



Chapter 5

Advanced Topics

5.1 Floor and Ceiling Functions

Definition 5.1.1 (Floor, Ceiling, and Fractional Part Functions)**.**

$$\lfloor x \rfloor = \text{Greatest integer less than or equal to x}$$

$$\lceil x \rceil = \text{Smallest integer greater than or equal to x}$$

$$\{x\} = \text{Fractional part of x (the value after the decimal point)}$$

Concept 5.1.2 (Common Floor and Ceiling Problems Techniques)

Most floor and ceiling problems can be solved using these techniques.

1. Make the substitution $x = \lfloor x \rfloor + \{x\}$
2. Use the floor or ceiling function to find an inequality
   For example, if you know that $y = \lfloor x \rfloor$, then $y \leq x < y + 1$
3. Graph your equations and look for intersection points (we recommend using graph paper)

5.2 Trivial Inequality

Theorem 5.2.1 (Trivial Inequality)
For real x, $x^2 \geq 0$
This means all perfect squares are 0 or more.

Corollary 5.2.2 (Completing the Square)
In a quadratic $Q(x) = ax^2 + bx + c$,

- If $a > 0$, then the minimum value of $Q(x)$ is $c - \frac{b^2}{4a}$ and occurs when $x = -\frac{b}{2a}$
- If $a < 0$, then the maximum value of $Q(x)$ is $c - \frac{b^2}{4a}$ and occurs when $x = -\frac{b}{2a}$

Remark 5.2.3
Simple, yet powerful. This is the core of all inequalities and how more advanced inequalities are derived.

The rest of the inequalities are optional for the AMC 10 but are still good to know.

5.3 AM-GM

Theorem 5.3.1 (AM-GM Inequality For 2 variables)
For non-negative reals a and b,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Basically, this means the average of 2 non-negative numbers (arithmetic mean) is always at least as big as the square root of the product of the 2 numbers (the geometric mean).

Note that equality in this expression occurs when $a = b$.

Corollary 5.3.2
The minimum value of $x + \frac{1}{x}$ is 2 and occurs when $x = 1$

- The minimum value of $a + b$ (if ab remains constant) occurs when $a = b$
- The maximum value of ab (if $a + b$ remains constant) occurs when $a = b$

Theorem 5.3.3 (AM-GM Inequality For More Variables)
For non-negative reals $a_1, a_2, \ldots a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_n}$$

Note that equality occurs when $a_1 = a_2 \cdots = a_n$. (essentially all the variables are equal).

Another way to say this is

$$\text{Average of n numbers} \geq \sqrt[n]{\text{product of all n numbers}}$$

Remark 5.3.4
This means in general,

$$\min(a_1 + a_2 + a_3 + \cdots + a_n) = n \cdot \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_n}$$

$$\max(a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_n) = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n$$

Essentially,

$$\min(\text{sum of all numbers}) = n \cdot \sqrt[n]{\text{product of all numbers})}$$

$$\max(\text{product of all numbers}) = (\text{average of all numbers})^n$$

Remark 5.3.5
Generally, we use AM-GM to maximize products or minimize sums.

Theorem 5.3.6 (Weighted AM-GM Inequality)
For non-negative reals, a_i, c_i,

$$\frac{c_1 \cdot a_1 + c_2 \cdot a_2 + \cdots + c_n \cdot a_n}{c_1 + c_2 + \cdots + c_n} \geq \sqrt[c_1+c_2+\cdots+c_n]{a_1^{c_1} \cdot a_2^{c_2} \cdot a_3^{c_3} \cdot \cdots \cdot a_n^{c_n}}$$

Remark 5.3.7
Weighted AM-GM is very similar to AM-GM. One way to visualize weighted AM-GM is that there are c_k number of terms which are all equal to a_k. So instead of writing

$a_k + a_k + \cdots + a_k$ c_k times in our sum we simply write $a_k \cdot c_k$, and instead of writing $a_k \cdot a_k \cdot \ldots$ c_k times in our product we simply write $a_k^{c_k}$.

Remark 5.3.8
We use weighted AM-GM when we are trying to make the sum of all terms a constant by multiplying weights to all (or some) the terms. Remember to divide by the weights you multiplied at the end.

5.4 Cauchy Schwarz

Theorem 5.4.1 (Cauchy Schwarz)
For reals a_i and b_i,

$$(a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n)^2 \leq (a_1^2 + a_2^2 + \ldots a_n^2)(b_1^2 + b_2^2 + \ldots b_n^2)$$

This means

$$(\text{the sum of the products of all } a_k \text{ and } b_k)^2 \leq$$
$$\text{product of the sum of squares of all } a_k \text{ and } b_k$$

Equality holds when the ratio of

$$\frac{a_i}{b_i}$$

for all i is the same.

Remark 5.4.2
If you ever forget which side the $\geq$ sign faces, just try a small example like $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, and $b_2 = 4$.

Remark 5.4.3
You generally want to apply Cauchy Schwarz when you are dealing with sums of squares.

Corollary 5.4.4 (Titu's Lemma)
For reals a_i and b_i,

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + b_3 + \cdots + b_n}$$

Alternately,
$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq \frac{(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n})^2}{b_1 + b_2 + b_3 + \cdots + b_n}$$

Note that this is a direct consequence of Cauchy Schwartz.

Concept 5.4.5 (Techniques For Optimization Problems)
Steps to find maximum/minimum of expressions

1. Try to find another simple expression for maximization is greater than or equal to the expression you are given OR minimization is less than the expression you are given by using 1 (or possibly even more) of the inequalities
   (a) Trivial Inequality
   (b) AM-GM
   (c) Weighted AM-GM (AM-GM weighted and unweighted are useful for maximizing products and minimizing sums)
   (d) Cauchy Schwartz (Cauchy Schwartz is useful when dealing with sums of squares)
2. Verify that the equality case of your inequality holds true with your problem conditions
3. Simplify your equality case and solve for the answer

5.5 Logarithms

Definition 5.5.1 (Logarithm)**.** A logarithm is the power to which a number must be raised in order to get some other number.
Logarithms are expressed as
$$a = log_b n$$
where b is the base and n is the number.
Basically, we are trying to calculate how many times we need to multiply the base to get the number a, or what power do we need to raise the base to get the number a.

Theorem 5.5.2 (Converting to Logarithm and Exponents)

$$log_x y = a \implies x^a = y$$

.

5.6 Logarithmic Formulas

Theorem 5.6.1 (Important Formulas)

$$\log_a a^r = r$$

$$\log_a bc = \log_a b + \log_a c$$

$$\log_a \frac{b}{c} = \log_a b - \log_a c$$

$$\log_a b^c = c \log_a b$$

$$\log_a b \cdot \log_b c = \frac{\log_b}{\log_a} \cdot \frac{\log_c}{\log_b} = \frac{\log_c}{\log_a} = \log_a c$$

$$\log_b a = \frac{1}{\log_a b}$$

$$\log_b a = \frac{\log_d a}{\log_d b}$$

Remark 5.6.2
This last formula is known as the ”Base Change Formula” and is the most useful of them all. Often times in logarithm problems you can just expand out your expression in terms of this formula and simplify the expression to get the answer.

Remark 5.6.3
These formulas are extremely important for working with logarithms and should definitely be memorized.

Remark 5.6.4
If you ever forget which way the sign of these logarithms are, you can just try a small example like $log_{10}100 + log_{10}1000 = log_{10}100,000$ so from here for example you could figure out the sum of logarithms identity.

Remark 5.6.5
Natural Logarithm (Base e) where e is the Euler's Number $= 2.71828$

A logarithm with a base of e is a called a natural log and is written as ln

Theorem 5.6.6 (Advanced Formulas)

$$\log_{a^m} a^n = \frac{n}{m}$$

$$\log_a \left(\frac{1}{b}\right) = -\log_a b$$

$$\log_{\frac{1}{a}} b = -\log_a b$$

Remark 5.6.7
These formulas are less important and aren't necessary for most logarithm problems, but still good to know.

Concept 5.6.8
When you take a logarithm of numbers which form a geometric progression, the logarithms of those numbers form an arithmetic progression.

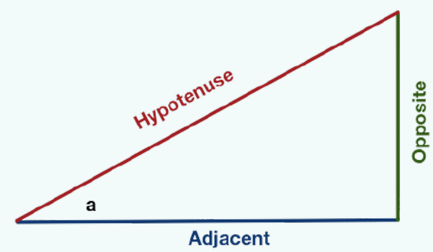
5.7 Trigonometric Identities

Theorem 5.7.1 (Trigonometric Identities)

$$\text{Sine: } \sin(a) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\text{Cosine: } \cos(a) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{Tangent: } \tan(a) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\sin(a)}{\cos(a)}$$



Remark 5.7.2
To remember the relationships, just use the mnemonics SOH, CAH, TOA:

SOH = Sin is Opposite over Hypotenuse

CAH = Cos is Adjacent over Hypotenuse

TOA = Tan is Opposite over Adjacent

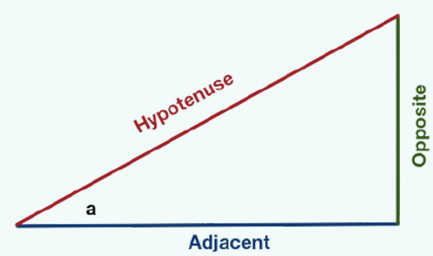
5.8 More Trigonometric Identities

Theorem 5.8.1 (More Trigonometric Identities)

$$\text{Cosecant: } \csc(a) = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{1}{\sin(a)}$$

$$\text{Secant: } \sec(a) = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{1}{\cos(a)}$$

$$\text{Cotangent: } \cot(a) = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{1}{\tan(a)} = \frac{\cos(a)}{\sin(a)}$$



5.9 Important Trigonometric Values

$\cos 0° = 1$	$\sin 0° = 0$
$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$
$\cos 30° = \frac{\sqrt{3}}{2}$	$\sin 30° = \frac{1}{2}$
$\cos 45° = \frac{\sqrt{2}}{2}$	$\sin 45° = \frac{\sqrt{2}}{2}$
$\cos 60° = \frac{1}{2}$	$\sin 60° = \frac{\sqrt{3}}{2}$
$\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$	$\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$
$\cos 0° = 1$	$\sin 0° = 0$
$\cos 90° = 0$	$\sin 90° = 1$
$\cos 120° = -\frac{1}{2}$	$\sin 120° = \frac{\sqrt{3}}{2}$
$\cos 135° = -\frac{\sqrt{2}}{2}$	$\sin 135° = \frac{\sqrt{2}}{2}$
$\cos 150° = -\frac{\sqrt{3}}{2}$	$\sin 150° = \frac{1}{2}$
$\cos 180° = 1$	$\sin 0° = 0$

5.10 Unit Circle Identities

Theorem 5.10.1 (Unit Circle Identities)

$$\sin(-a) = -\sin(a)$$

$$\sin(a) = \sin(180 - a)$$

$$\cos(a) = \cos(-a)$$

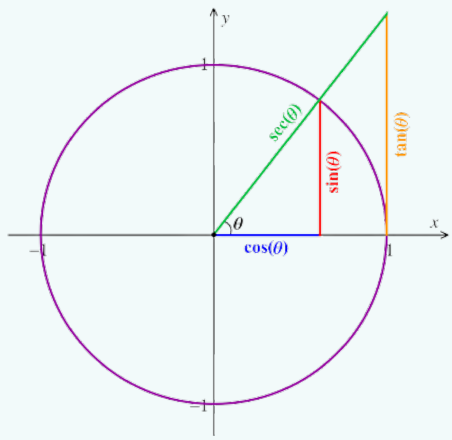
$$\cos(a) = -\cos(180 - a)$$

$$\tan(a) = -\tan(180 - a)$$

$$\tan(-a) = -\tan(a)$$

5.11 Pythagorean Identities

Theorem 5.11.1 (Pythagorean Identities)



$$\sin^2(a) + \cos^2(a) = 1$$

$$\tan^2(a) + 1 = \sec^2(a)$$

$$\cot^2(a) + 1 = \csc^2(a)$$

5.12 Double Angle Identities

Theorem 5.12.1 (Double Angle Identities)

$$\sin(2a) = 2\sin(a)\cos(a)$$

$$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$$

$$\tan(2a) = \frac{2\tan(a)}{1 - \tan^2(a)}$$

5.13 Addition and Subtraction Identities

Theorem 5.13.1 (Addition and Subtraction Identities)

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

$$\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan a \tan b}$$

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 + \tan a \tan b}$$

5.14 Half Angle Identities

Theorem 5.14.1 (Half Angle Identities)

$$\sin\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1-\cos(a)}{2}}$$

$$\cos\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1+\cos(a)}{2}}$$

5.15 Sum to Product Identities

Theorem 5.15.1 (Sum to Product Identities)

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

5.16 Product to Sum Identities

Theorem 5.16.1 (Product to Sum Identities)

$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$$

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

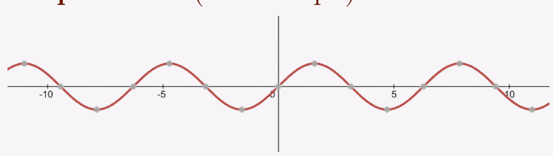
5.17 Graphing to Solve Trigonometry Problems

Definition 5.17.1. A periodic function is a trigonometric function which repeats a pattern of y-values at regular intervals. One complete repetition of the pattern is called a cycle. The period of a function is the horizontal length of one complete cycle.

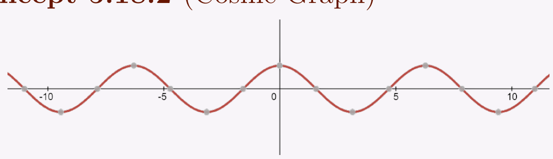
Period of sin, cos, and tan is 2π

5.18 Periods and Graphs of Trigonometric Functions

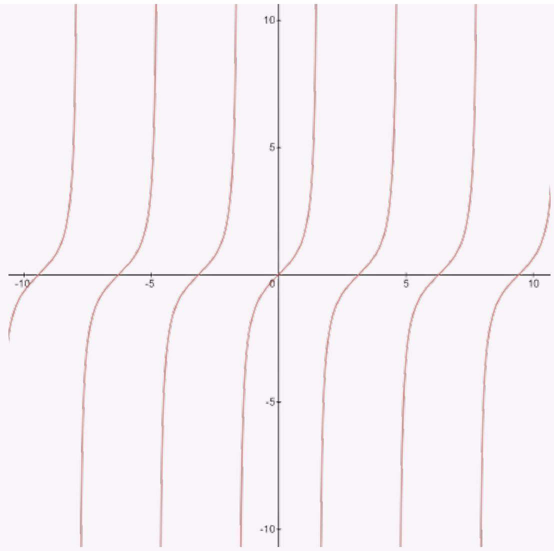
Concept 5.18.1 (Sine Graph)



Concept 5.18.2 (Cosine Graph)



Concept 5.18.3 (Tan Graph)

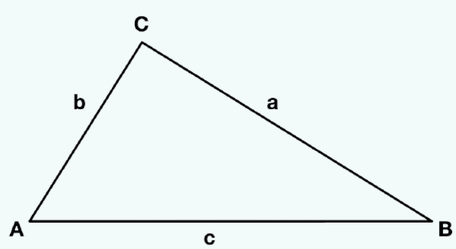


Remark 5.18.4
Long trigonometric expressions can be evaluated by telescoping, using identities in clever ways, complex number substitutions (see complex numbers section below).

5.19 Area of a Triangle using Trigonometry

Theorem 5.19.1 (Area of a Triangle using trigonometry)
In a triangle with side lengths, a, b, c, where the angle between sides a and b is denoted by C

$$\text{Area of the triangle} = \frac{1}{2} \cdot ab \cdot \sin(C)$$

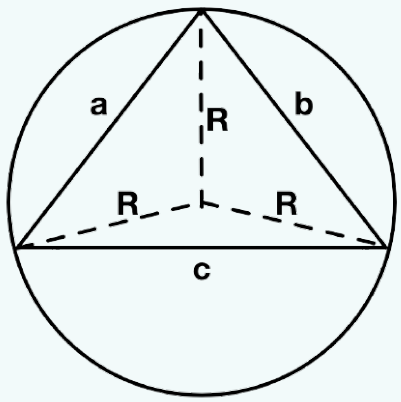


5.20 Law of Sines

Theorem 5.20.1 (Law of Sines)
In a triangle with sides a, b, c and angles A, B, and C where the side a is opposite to the angle A, the side b is opposite to the angle B, and the side c is opposite the angle C, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

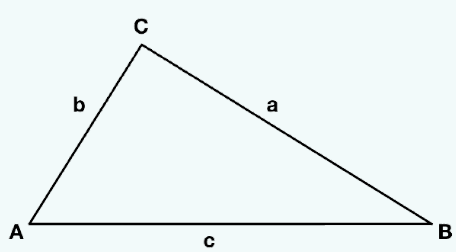
where R is the circumradius of the triangle.



5.21 Law of Cosines

Theorem 5.21.1 (Law of Cosines)
In a triangle with side lengths, a, b, c, where the angle between sides a and b is denoted by C

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$



Video Lectures

Solving area problems using trigonometry

5.22 Complex Numbers

Definition 5.22.1. A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the "imaginary unit". a is the real part of our number, and bi is the imaginary part. Complex numbers are often represented by the variable z.

Definition 5.22.2. $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Remark 5.22.3
Powers of i cycle every 4 terms, so $i^{4n} = i^4 = 1, i^{4n+1} = i = \sqrt{-1}, i^{4n+2} = i^2 = -1, i^{4n+3} = i^3 = -i$

Theorem 5.22.4 (Adding Complex Numbers)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Theorem 5.22.5 (Subtracting Complex Numbers)

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Theorem 5.22.6 (Multiplying Complex Numbers)

$$(a + bi) \cdot (c + di) = (ac - bd) + (bc + ad)i$$

Definition 5.22.7 (Real and Imaginary Parts)**.** The imaginary part of a complex number $a + bi$ is b and the real part is a.

Remark 5.22.8
The imaginary part does not include a factor of i.

Definition 5.22.9. A complex conjugate is found by flipping the sign of the imaginary part of complex number, and is represented as $\bar{z}$.

Theorem 5.22.10 (Finding Conjugates)

$$\overline{z} = \overline{a + bi} = a - bi$$

Definition 5.22.11 (Complex Plane)**.** A complex number can also be represented geometrically by expressing $a + bi$ as (a, b) on the Complex plane. The x-axis represents the Real axis and the y-axis represents the Imaginary axis.

Definition 5.22.12. The magnitude of a complex number is represented by $|z|$, and is the distance of a complex number (a, b) from the origin.

Theorem 5.22.13 (Magnitude of a Complex Number)

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

Theorem 5.22.14 (Multiplying Complex Numbers with their Conjugates)

$$(a + bi)(a - bi) = a^2 + b^2$$

Theorem 5.22.15
For a complex number z, $z \times \overline{z} = \bar{z}^2$

Remark 5.22.16
This is derived from the theorem above. It is super useful!

5.22.1 Polar Form

Definition 5.22.17. The angle that the positive real axis makes with the ray that connects the origin with a complex number is called the argument of that complex number and is represented by θ.

Theorem 5.22.18 (Argument of a Complex Number)
The argument θ of a complex number $a + bi$ is

$$\tan\theta = \frac{b}{a}$$

.

Definition 5.22.19. The distance between 0 and a complex number is sometimes called the modulus of that complex number and is represented by r.

Theorem 5.22.20 (Modulus of a Complex Number)
The modulus r of a complex number $a + bi$ is

$$r = |a + bi| = \sqrt{a^2 + b^2}$$

.

Definition 5.22.21. Polar form is another way to represent a complex number based on its modulus r and argument θ.

Theorem 5.22.22 (Polar Form)

$$z = a + bi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$$

Remark 5.22.23
$\operatorname{cis}\theta$ is just short for $\cos\theta + i\sin\theta$

Remark 5.22.24
Trigonometric ratios tell us that $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$, which we can rearrange to see that $r\cos\theta = a$ and $r\sin\theta = b$. Plugging in these values gives us the polar form formula.

Remark 5.22.25
$\cos\theta + i\sin\theta$ can also be written as $\operatorname{cis}\theta$.

Theorem 5.22.26 (Euler's Formula)
Euler's Formula tells us that

$$\cos\theta + i\sin\theta = e^{i\theta}$$

, which tells us that

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

.

Remark 5.22.27
Euler's Identity is a special case of Euler's Formula and tells us that

$$e^{\pi i} = -1$$

.

Definition 5.22.28. Roots of unity are the complex solutions to an equation $x^n = 1$, for some positive integer n. There will always be n solutions to $x^n = 1$.

Theorem 5.22.29 (Roots of Unity)
The set of the nth roots of unity is

$$e^{2k\pi i/n}$$

for

$$k \in \{1, 2, \ldots n\}$$

.

Theorem 5.22.30 (Rotating a Point)
To rotate a point θ radians counterclockwise, covert a coordinate to its corresponding complex number and multiply it by $e^{i\theta}$. Converting this back to ordered pairs gives us our answer.

Theorem 5.22.31 (De Moivre's Theorem)
For a complex number $z = re^{i\theta}$ and a real number n,

$$z^n = (re^{i\theta})^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

Remark 5.22.32
We can use this to evaluate expressions like

$$(\sqrt{3} + i)^8$$

much easier because we just convert to polar form and apply De Moivre's Theorem.

Remark 5.22.33
DeMoivre's Theorem is very useful when dealing with complex numbers and exponents.

Concept 5.22.34
Complex numbers and their relations to circles makes them easy to work with for many geometry problems, especially when dealing with polygons such as equilateral triangles or squares.

How to solve geometry problems using complex numbers:

1. Assign a complex number to 1 or more of the coordinates
2. To find the complex numbers for other points, multiple/divide by $e^{i\theta}$
3. Use the information you have to solve for what you are asked in the problem

Remark 5.22.35
We can also view algebraic complex number problems geometrically.

5.23 Complex Numbers in Polynomials

Definition 5.23.1. A polynomial of degree n has n roots, and these roots may be complex. For binomials, if our discriminant is negative we have complex roots.

5.24 Complex Numbers in Trigonometry

Theorem 5.24.1 (Sin and Cos values in terms of complex numbers)

$$\sin\theta = \frac{e^{i\theta} + e^{i(180-\theta)}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{i(-\theta)}}{2}$$

$$\tan\theta = \frac{e^{i\theta} + e^{i(180-\theta)}}{e^{i\theta} + e^{i(-\theta)}} i$$

Remark 5.24.2

By using these substitutions, we can bash out the value of trigonometric expressions easily without clever manipulation of trigonometric identities that would be needed to solve the problem otherwise.