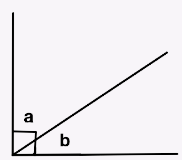
Chapter 1

Geometry

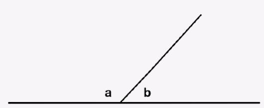
Concept 1.0.1 (Complementary Angle)

Complementary angles are a pair of angles with the sum of 90 degrees



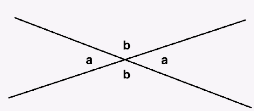
Concept 1.0.2 (Supplementary Angle)

Supplementary angles are a pair of angles with the sum of 180 degrees



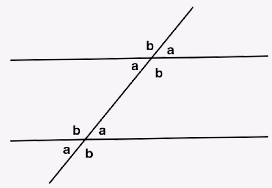
Concept 1.0.3 (Intersecting lines)

When two lines intersect, the vertical angles are equal. Vertical angles are each of the pairs of opposite angles made by two intersecting lines. "Vertical" in this case means they share the same Vertex (corner point), not the usual meaning of up-down.



Concept 1.0.4 (Parallel Lines)

Corresponding angles equal

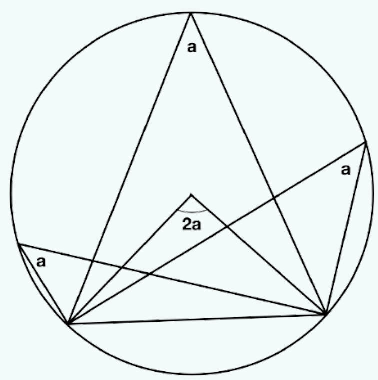


Remark 1.0.5

Using this property along with supplementary angles, we can derive many other angle relations.

Theorem 1.0.6 (Inscribed Arc Theorem)

The angle formed by an arc in the center or the arc angle is double of the angle formed on the edge.



Remark 1.0.7

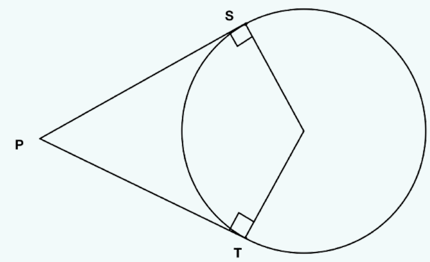
Circles are really useful for angle chasing so keep an eye out for the inscribed arc theorem that can be used in many angle chasing problems.

Remark 1.0.8

A useful trick to solving angle chasing problems with regular polygons is to draw a circle around the polygon and use the inscribed arc theorem.

Theorem 1.0.9 (Right Angle Tangency Point)

If you connect the center of a circle to the point where the circle and a line are tangent, they will form a right angle.



1.1 Polygons

Theorem 1.1.1

$$\text{Sum of interior angle of a polygon} = (n-2)\cdot 180$$

$$\text{Interior angle of a regular polygon} = \frac{(n-2)}{n}\cdot 180$$

$$\text{Exterior angle of a regular polygon} = \frac{360}{n}$$

Fact 1.1.2. Important Interior Angles

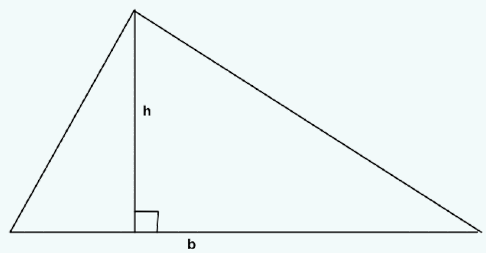
Number of sides in regular polygon	Interior Angle of regular polygon
3	60
4	90
5	108
6	120
8	135
9	140
10	144

1.2 Area of a Triangle

There are many ways to calculate the area of a triangle. Here are some of the most useful formulas for calculating the area of a triangle:

Theorem 1.2.1 (Using base and height)
A triangle with base b and height h has an area of

$$\frac{1}{2} \cdot b \cdot h$$



Theorem 1.2.2 (Heron's Formula)
A triangle with sides a, b, c and semiperimeter s has an area of

$$\sqrt{s(s-a)(s-b)(s-c)}$$

Definition 1.2.3 (Inradius)**.** The inradius of a triangle is the radius of the inscribed circle in the triangle.

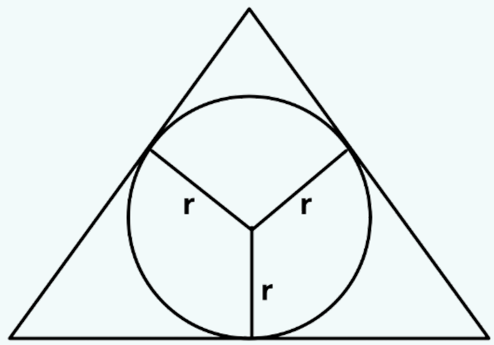
Definition 1.2.4 (Incenter)**.** The incenter of a triangle is the intersection of all the angle bisectors. This point is also the center of the incircle, and equidistant from all the three sides.

Theorem 1.2.5 (Area of a triangle using inradius)

A triangle with inradius r (the radius of the circle that can be inscribed in a triangle) and semiperimeter s has an area of:

$$\text{inradius} \cdot \text{semiperimeter}$$

$$rs$$



Remark 1.2.6

Note that if we know the area of the triangle and it's semi-perimeter, we can apply the inradius formula to find the inradius of the triangle.

Theorem 1.2.7

Inradius r of a right triangle:

$$r = \frac{1}{2}(a + b - c)$$

where a and b are the legs of the triangle, and c is the hypotenuse.

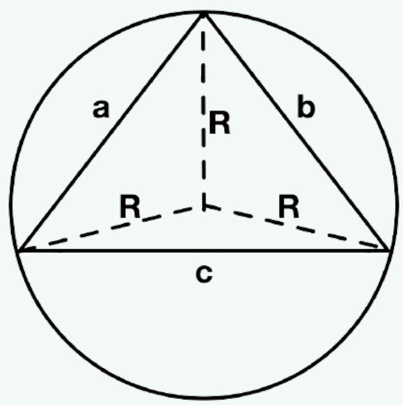
Definition 1.2.8 (Circumradius)**.** The circum-radius of a triangle is the radius of circle that a triangle is inscribed in.

Definition 1.2.9 (Circumcenter)**.** The circumcenter of a triangle is the intersection of all 3 perpendicular bisectors (the line that bisects a segment and is perpendicular to it). This point is also the center of the circumcircle and equidistant from all the three vertices.

Theorem 1.2.10 (Area of a triangle using circumradius)

A triangle with circumradius R (the radius of the circle that the triangle can be inscribed in) and sides a, b, c has an area of

$$\frac{abc}{4R}$$



Remark 1.2.11

Similar to the inradius problem, if we know all 3 sides of a triangle, we can apply Heron's and easily calculate the circumradius of the triangle.

1.3 Special Triangles

1.3.1 Equilateral Triangle

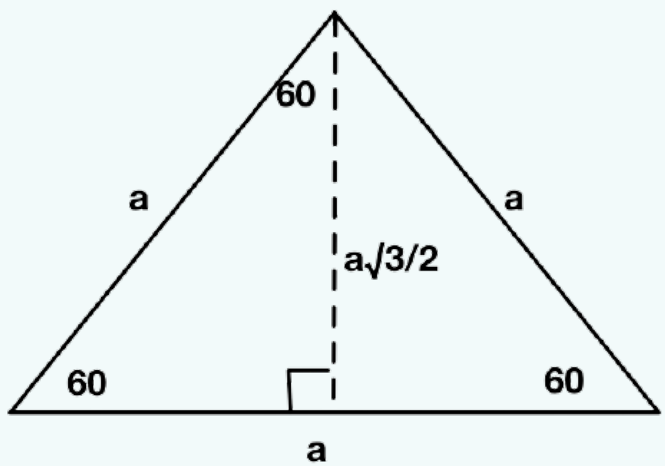
Theorem 1.3.1

If the side length of an equilateral triangle is a

$$\text{Height of the triangle} = \frac{\sqrt{3}}{2}a$$

This follows directly from the $30 - 60 - 90$ triangle.

$$\text{Area of the triangle} = \frac{\sqrt{3}}{4}a^2$$



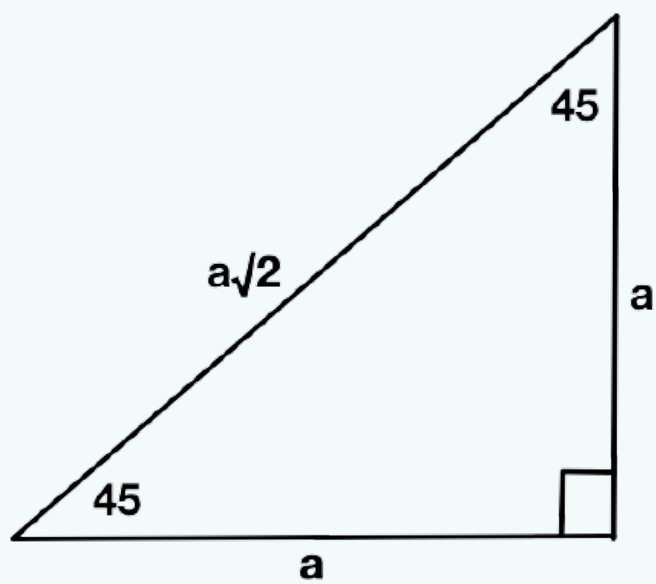
1.3.2 45-45-90 Triangle

Theorem 1.3.2

If the side length of a 45-45-90 triangle is a

$$\text{hypotenuse of the triangle} = \sqrt{2} \times \text{side length} = \sqrt{2}a$$

$$\text{Area of the triangle} = \frac{1}{2} \times \text{side length}^2 = \frac{1}{2}a^2$$



1.3.3 30-60-90 Triangle

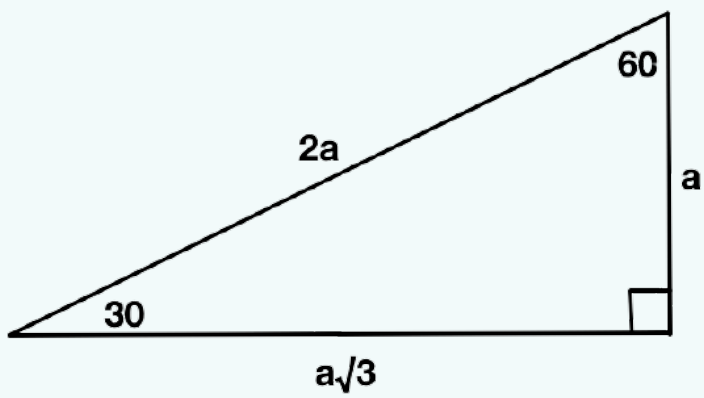
Theorem 1.3.3

If the short leg length of a 30-60-90 triangle is a

$$\text{Long Leg of the triangle} = \sqrt{3} \times \text{short leg} = \sqrt{3}a$$

$$\text{hypotenuse of the triangle} = 2 \times \text{short leg} = 2a$$

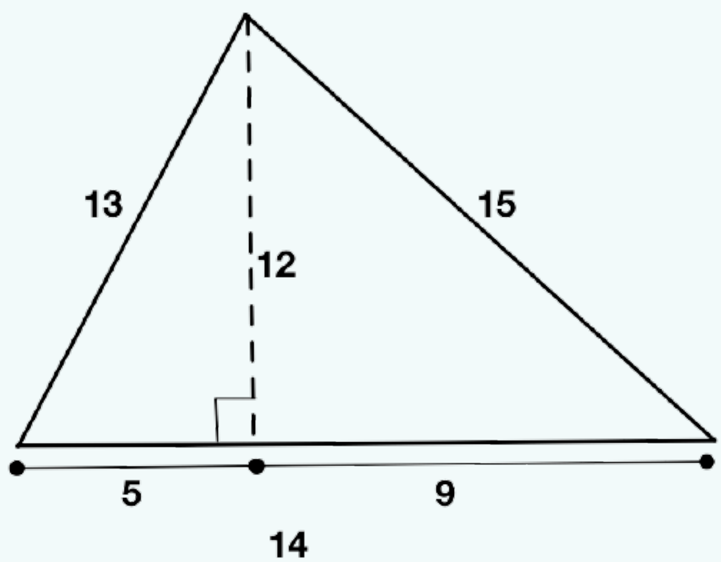
$$\text{Area} = \frac{\sqrt{3}}{2} \times \text{short leg}^2 = \frac{\sqrt{3}}{2}a^2$$



1.3.4 13-14-15 Triangle

Theorem 1.3.4

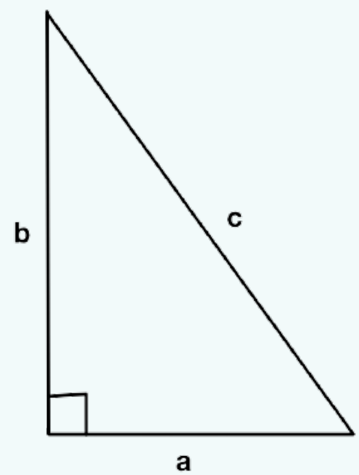
If the three sides of a triangle are 13, 14, and 15, it can be divided into two right triangles with side lengths:
5, 12, 13 and 9, 12, 15
Area of this triangle $= \frac{1}{2} \times 14 \times 12 = 84$



1.4 Pythagorean Theorem

Theorem 1.4.1 (Pythagorean Theorem)
A right triangle with legs a and b and hypotenuse c satisfies the following relation:

$$c^2 = a^2 + b^2$$



Fact 1.4.2. Important Pythagorean Triples
3, 4, 5
5, 12, 13
7, 24, 25
8, 15, 17
9, 40, 41
20, 21, 29

If all numbers in a pythagorean triple are multiplied by a constant, the resulting numbers still form a pythagorean triple.

For example: These are all pythagorean triples:
3, 4, 5
6, 8, 10
9, 12, 15
12, 16, 20
15, 20, 25

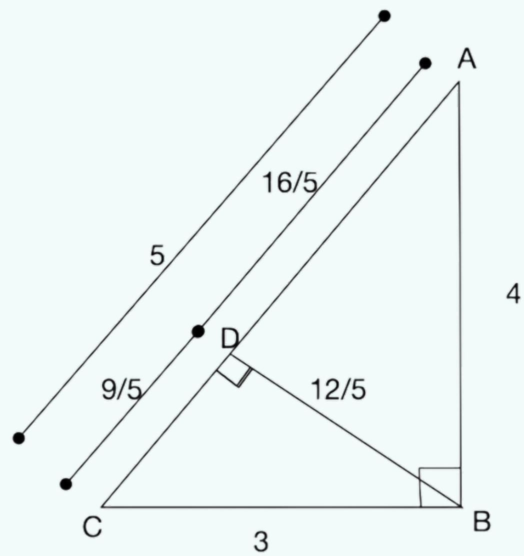
Theorem 1.4.3 (Special Properties of right triangles)
In a right triangle ABC where B is the right angle, the following triangles are similar

$$\triangle ABC \sim \triangle ADB \sim \triangle BDC$$

Length of the perpendicular to the hypotenuse (BD) $= \frac{AB \cdot BC}{AC}$

Also note that:

$$\mathrm{AD} \cdot \mathrm{CD} = \mathrm{BD}^2$$
$$\mathrm{AD} \cdot \mathrm{AC} = \mathrm{AB}^2$$
$$\mathrm{CD} \cdot \mathrm{CA} = \mathrm{CB}^2$$



1.5 Triangle Properties

Definition 1.5.1 (Cevian)**.** A cevian is any line from any vertex of a triangle to the opposite side. Medians and angle bisectors are special cases of cevians.

Definition 1.5.2 (Median)**.** A median is a line connecting a point to the midpoint of the opposite side.

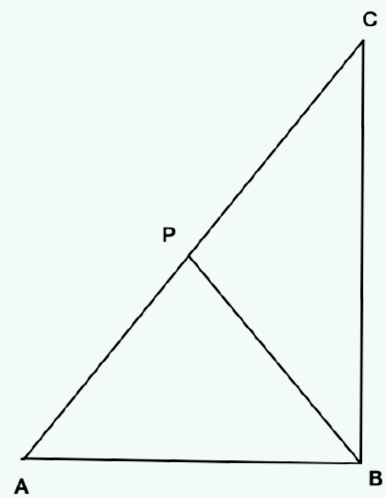
Definition 1.5.3 (Centroid)**.** In a triangle, the intersection of all 3 medians in a triangle is the centroid.

Theorem 1.5.4

The centroid of a triangle is on the median and it is $\frac{2}{3}$ of the way from from one of vertices to the midpoint of the opposite side.

Theorem 1.5.5 (Median in a Right Triangle)

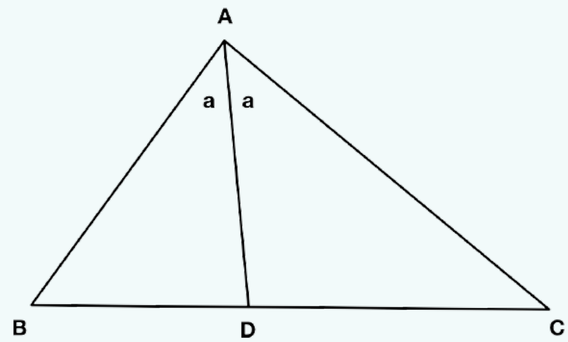
In a right triangle ABC, let the median from point B intersect AC at a point P. Then AP = BP = CP.
Basically, in a right triangle AC is the diameter of the circumcircle, and PC, PA, and PB are radii of the circumcircle.



1.6 Angle Bisector Theorem

Theorem 1.6.1 (Angle Bisector Theorem)
If the line AD bisects angle A, then

$$\frac{AB}{BD} = \frac{AC}{CD}$$



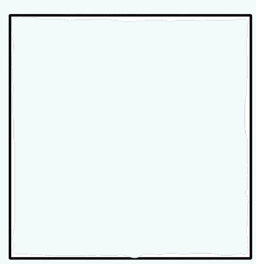
1.7 Square

Theorem 1.7.1 (Area of a Square)
Any square with side length s has an area of

$$s^2$$

and a perimeter of

$$4s$$



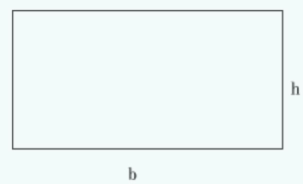
1.8 Rectangle

Theorem 1.8.1 (Area of a Rectangle)
Any rectangle with base b and height h has an area of

$$bh$$

and a perimeter of

$$2b + 2h$$



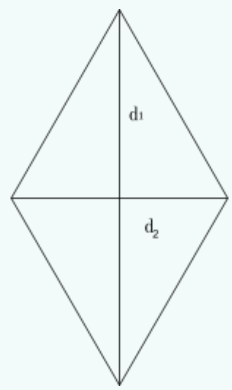
1.9 Rhombus

Theorem 1.9.1 (Area of a Rhombus)
A rhombus with diagonals d_1 and d_2 has an area of

$$\frac{1}{2}d_1d_2$$

and a perimeter of

$$2 \times \sqrt{d_1^2 + d_2^2}$$

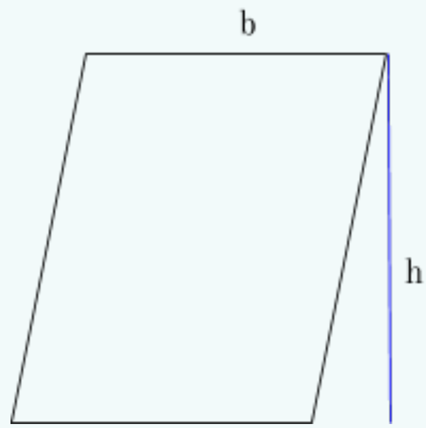


1.10 Parallelogram

Theorem 1.10.1 (Area of a Parallelogram)

A parallelogram with base b and height h has an area of

$$bh$$

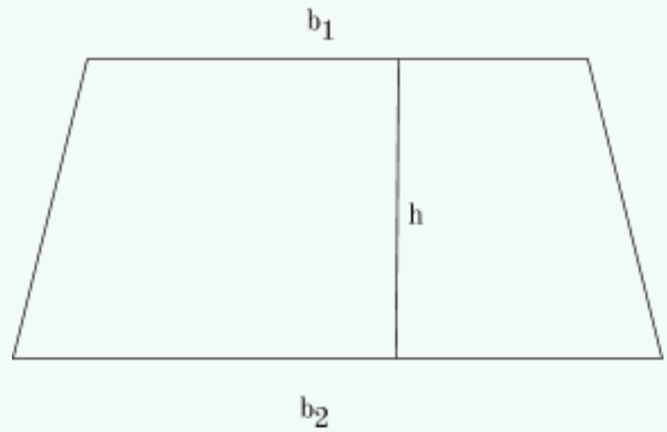


1.11 Trapezoid

Theorem 1.11.1 (Area of a Trapezoid)

A trapezoid with 2 bases b_1 and b_2 and a height h has an area of

$$\frac{b_1 + b_2}{2} \cdot h$$



1.12 Circle Properties

Theorem 1.12.1 (Area and Circumference)
A circle with radius r has

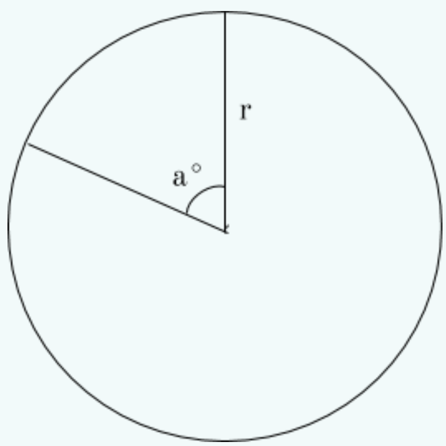
$$\text{Area} = \pi r^2$$

$$\text{Circumference} = 2\pi r$$

Theorem 1.12.2 (Arcs of a circle)
An arc of a circle with radius r and angle $a°$

$$\text{Area of a sector} = \pi r^2 \times \frac{a°}{360} = \pi \times \text{radius}^2 \times \text{fraction of circle in sector}$$

$$\text{Length of the arc} = 2\pi r \times \frac{a°}{360} = 2\pi \times \text{radius} \times \text{fraction of circle in sector}$$



Definition 1.12.3 (Angle of an arc)**.** This is the angle that the arc makes at the center of the circle.

1.13 Similar Triangles

Theorem 1.13.1

For congruent triangles:

1. All the angles of the triangles are same
2. All corresponding sides are equal
3. Area, perimeter, inradius, circumradius, etc. are equal

Concept 1.13.2 (Congruence Test)

Two triangles are congruent if the three angles and sides in the triangle are the same. In other words, the triangles are the same (not necessarily same orientation). In general, triangles are congruent if:

- AAS congruence: Two angles of the triangles are same and the 1 side next to the 2 angles are equal
- SAS Congruence (Side Angle Side): Two sides are equal and the angle between the sides are equal
- SSS congruence (Side Side Side): All three sides are congruent
- HL congruence (Hypotenuse Leg): In a right triangle, hypotenuse and leg are equal
- LL congruence (LL Leg): In a right triangle, the two legs are equal

Warning: SSA is not a valid triangle congruence: if 2 corresponding sides are congruent and the angle which is not between the 2 congruent sides is equal, then the triangles are not necessarily congruent

1.14 Similar Triangles

Concept 1.14.1 (Similarity Test)

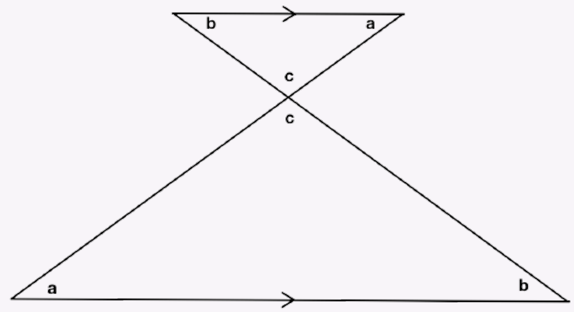
Two triangles are similar if the three angles in the triangle are the same. In other words, the triangles are the same shape multiplied by a scale factor.

In general, triangles are similar if:

- AA similarity: Two angles of the triangles are same, which basically means that the third angle will be equal)
- SAS similarity (Side Angle Side): Two sides are proportional and the angle between the sides is equal
- SSS similarity (Side Side Side): All three sides are proportional
- HL similarity (Hypotenuse Leg): In a right triangle, the hypotenuse and leg are proportional
- LL similarity (LL Leg): In a right triangle, the two legs are proportional

Warning: SSA does not mean triangles are similar

An easy way to detect similar triangles is if bases of triangles are parallel and the sides of the triangles are collinear (see figure below)



Theorem 1.14.2

For similar triangles:

1. All the angles of the triangles are same
2. All corresponding sides have same ratio
3. Area ratio is the square of side length ratio

1.15 Hexagon

Theorem 1.15.1

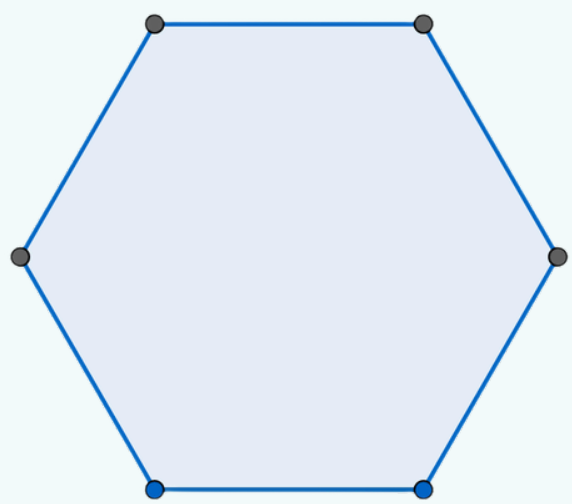
$$\text{Sum of interior angle of a regular hexagon} = (6-2) \cdot 180 = 720$$

$$\text{Interior angle of a regular hexagon} = \frac{(6-2)}{6} \cdot 180 = 120$$

$$\text{Exterior angle of a regular hexagon} = \frac{360}{6} = 60$$

$$\text{Area of a regular hexagon} = 6 \cdot \frac{\sqrt{3}}{4} s^2$$

$$\text{Length of the diagonal of a regular hexagon} = 2s$$



Remark 1.15.2
A regular hexagon can be divided into 6 congruent equilateral triangles.

1.16 Octagon

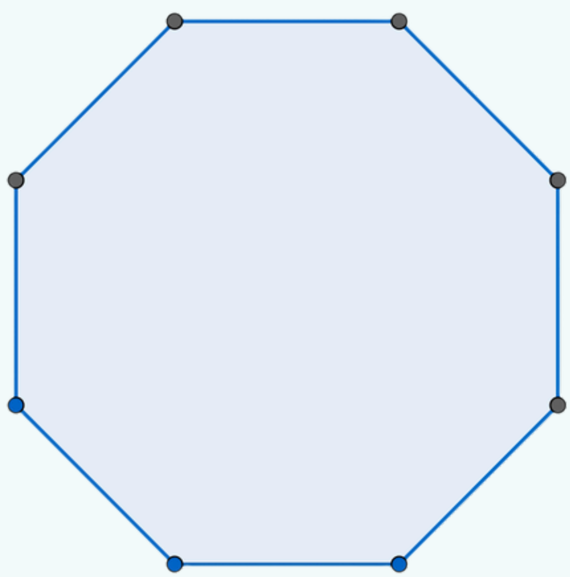
Theorem 1.16.1

$$\text{Sum of interior angle of a regular octagon} = (8-2)\cdot 180 = 1080$$

$$\text{Interior angle of a regular octagon} = \frac{(8-2)}{8}\cdot 180 = 135$$

$$\text{Exterior angle of a regular octagon} = \frac{360}{8} = 45$$

$$\text{Area of a regular octagon} = 2(1+\sqrt{2})s^2$$



1.17 Length of complex shapes

Concept 1.17.1
Finding Length of Complex Shapes

- Having equal angles means equal lengths and vice versa
- Be on the lookout for 90 degree angles, as you can use Pythagorean theorem
- Split the length into multiple components by using some of these techniques
  - Drawing extra lines
  - Dropping Altitudes
- Extending lines to create similar triangles, special triangles, etc. and then subtracting the extra length

1.18 Hexagon

Theorem 1.18.1

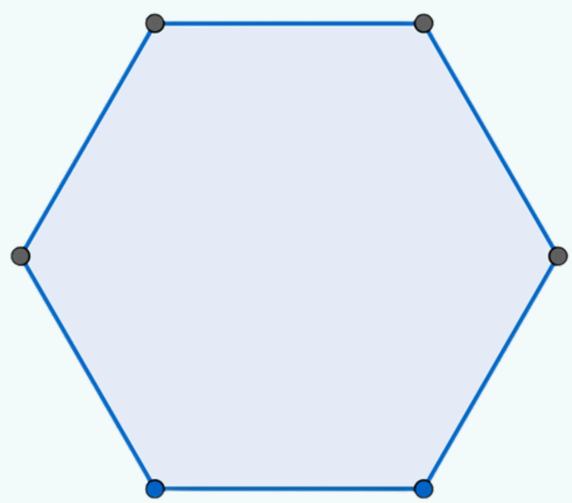
$$\text{Sum of interior angle of a regular hexagon} = (6-2)\cdot 180 = 720$$

$$\text{Interior angle of a regular hexagon} = \frac{(6-2)}{6}\cdot 180 = 120$$

$$\text{Exterior angle of a regular hexagon} = \frac{360}{6} = 60$$

$$\text{Area of a regular hexagon} = 6\cdot\frac{\sqrt{3}}{4}s^2$$

$$\text{Length of the diagonal of a regular hexagon} = 2s$$

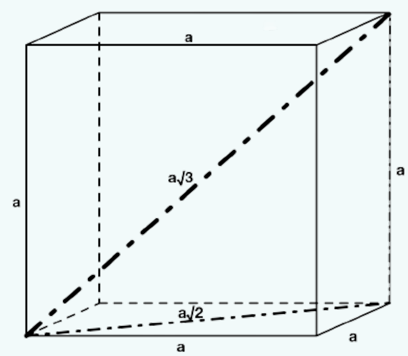


1.19 Cube

Theorem 1.19.1 (Volume and Surface Area of a cube)

$$\text{Volume of a cube} = (\text{side length})^3 = a^3$$
$$\text{Surface area of a cube} = 6 \times (\text{side length})^2 = 6a^2$$

$$\text{Length of space diagonal of a cube} = \sqrt{3} \times \text{side length} = \sqrt{3}a$$



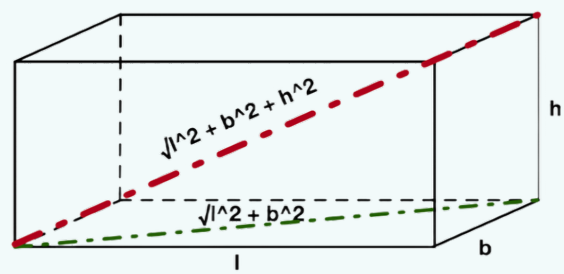
1.20 Prism

Theorem 1.20.1 (Volume and Surface Area of a rectangular prism)

$$\text{Volume of a rectangular prism} = l \times b \times h = \text{product of all three dimensions}$$

$$\text{Surface area of a rectangular prism} = 2(lb + bh + lh)$$

$$\text{Length of space diagonal of a rectangular prism} = \sqrt{l^2 + b^2 + h^2}$$



Theorem 1.20.2 (Volume of a Prism)

$$\text{Volume of any prism} = \text{base area} \cdot \text{height}$$

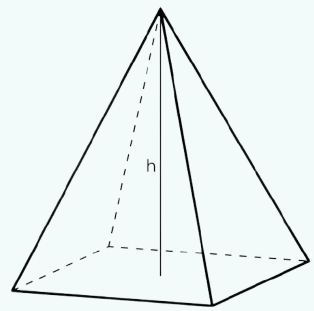
Theorem 1.20.3 (Surface Area of a Prism)

$$\text{Volume of any prism} = 2 \cdot \text{base area} + \text{base perimeter} * \text{height}$$

1.21 Pyramid

Theorem 1.21.1 (Volume of a Pyramid)

$$\text{Volume of any pyramid} = \frac{1}{3} \cdot \text{base area} \cdot \text{height}$$



Theorem 1.21.2 (Volume of a Regular Square Pyramid (all sides equal))
When the pyramid has a square base, and all the sides are equal

$$\text{Volume of a regular pyramid} = \frac{\sqrt{2}}{6} s^3$$

Theorem 1.21.3 (Volume of a Regular Tetrahedron (all sides equal))

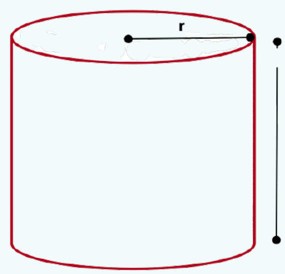
$$\text{Volume of a regular tetrahedron} = \frac{\sqrt{2}}{12} s^3$$

1.22 Cylinder

Theorem 1.22.1 (Volume and Surface Area of a cylinder)

$$\text{Volume of a cylinder} = \pi r^2 h$$

$$\begin{aligned} \text{Surface area of a cylinder} &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r(r+h) \end{aligned} \tag{1.1}$$



1.23 Cone

Theorem 1.23.1 (Volume and Surface Area of a cone)

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h$$

which basically means

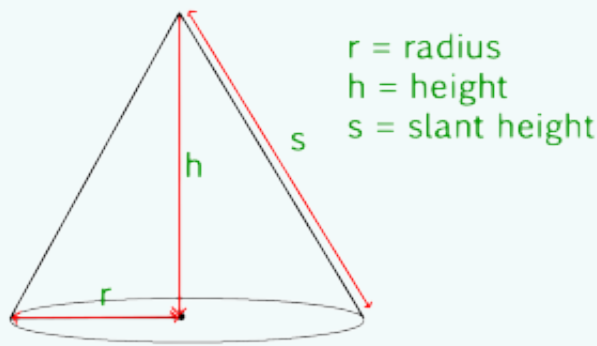
$$\text{Volume of a Cone} = \frac{1}{3}\pi \cdot \text{radius}^2 \cdot \text{height}$$

$$\text{Surface area of a cone} = \pi r^2 + \pi r s = \pi r(r+s)$$

where s is the lateral or slant height

which can also be written as

$$\pi \cdot \text{radius}^2 + \pi \cdot \text{radius} \times \text{slant height}$$



Remark 1.23.2
The slant height s can be calculated by the following formula

$$s = \sqrt{r^2 + h^2}$$

or

$$\text{slant height} = \sqrt{\text{radius}^2 + \text{height}^2}$$

Concept 1.23.3 (Cone Unfolding)
The sides of a cone can be unfolded to form a sector a circle as well. The circumference of the base of the cone is equal to the length of the arc of the sector and the slant height

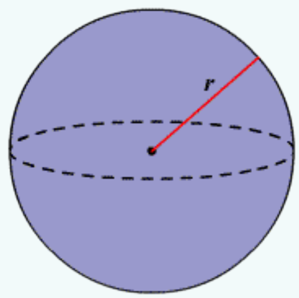
of the cone is equal to the radius of the sector.

1.24 Sphere

Theorem 1.24.1 (Volume and Surface Area of a sphere)

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

$$\text{Surface area of a sphere} = 4\pi r^2$$



Chapter 2

Number Theory

2.1 Primes

Definition 2.1.1 (Primes)**.** Primes are numbers that have exactly two factors: 1 and the number itself.
Example: 2, 3, 5, 7, 11, 13, 17, 19, 23 are all primes
Note: 1 is not a prime and 2 is the only even prime.

Remark 2.1.2
It's highly recommended to memorize first few primes as this can save some time during the contest: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37

Concept 2.1.3 (Divisibility Rules)

2	Last digit is even
3	Sum of digits is divisible by 3
4	Last 2 digits divisible by 4
5	Last digit is 0 or 5
6	Divisible by 2 and 3
7	Take out factors of 7 until you reach a small number that is either divisible or not divisible by 7
8	Last 3 digits are divisible by 8
9	Sum of digits is divisible by 9
10	Last digit is 0
11	Calculate the sum of odd digits (O) and even digits (E). If $\lvert O-E\rvert$ is divisible by 11, then the number is also divisible by 11
12	Divisible by 3 and 4
15	Divisible by 3 and 5

2.2 Prime Factorization

Prime factorization is a way to express each number as a product of primes.

Examples:

The prime factorization of 21 is 3×7

The prime factorization of 60 is $2^2 \times 3 \times 5$

Theorem 2.2.1 (Legendre's Theorem)

$$v_p(n!) = \text{Number of powers of } p \text{ in } n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$$

Remark 2.2.2
This formula seems complicated, but it much easier to apply than it may seem. This basically means

$$\text{number of factors of p in } n! =$$

$$\text{the number of multiples of p} \leq n + \text{the number of multiples of } p^2 \leq n + \dots$$

. See the video above for an easier explanation.

2.3 Factors

Theorem 2.3.1 (Number of Factors of a Number)
If the prime factorization of the number is expressed as:

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the number of factors of this number is

$$(e_1 + 1)(e_2 + 1) \dots (e_k + 1)$$

Concept 2.3.2
Basically, in order to find the number of factors of a number:

1. Find the prime factorization of the number
2. Add 1 to all of the exponents
3. Multiply them together

2.4 Sum of Factors

Theorem 2.4.1 (Sum of Factors of a Number)
If the prime factorization of the number is expressed as:

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the number of factors of this number is

$$(1+p_1^1+p_1^2+\cdots+p_1^{e_1-1}+p_1^{e_1})(1+p_2^1+p_2^2+\cdots+p_2^{e_2-1}+p_2^{e_2})\ldots(1+p_k^1+p_k^2+\cdots+p_k^{e_k-1}+p_k^{e_k})$$

Concept 2.4.2 (Sum of Factors of a Number)
Essentially, for a prime p in the prime factorization, first find the sum of p^k for all possible exponents k in the prime factorization. Then, we will multiply all such sums for all of the primes to get the sum of all the factors of the number. This may seem complicated, but it will make more sense with an example

Theorem 2.4.3 (Product of Factors of a Number)
The product of factors of a number n where it has f factors (this can be calculated using the number of factors formula) is $n^{\frac{f}{2}}$

2.5 GCD & LCM

Theorem 2.5.1

The product of GCD and LCM of two numbers is equal to the product of the two numbers:

$$GCD(m,n) \times LCM(m,n) = m \times n$$

Remark 2.5.2

Let's explore the reason why this is true. Let's say that m has p^a in it's prime factorization, and n has p^b in it's prime factorization. Then, the product mn will have p^{a+b} in it's prime factorization.

Remember from above how the gcd can be found by taking the lowest prime exponents and the lcm can be found by taking the highest prime exponents.

Therefore, if a is smaller than b, than the gcd will have p^a in it's prime factorization, and the lcm will have p^b in it's prime factorization. On the other hand, if b is smaller than a, than the gcd will have p^b in it's prime factorization, and the lcm will have p^a in it's prime factorization.

Therefore, no matter what, the product of the gcd and lcm will have p^{a+b} in it's prime factorization. This is true for all primes, so the products are equal.

Theorem 2.5.3

$$\gcd(ac, bc) = c \cdot \gcd(a, b)$$

Theorem 2.5.4

$$\operatorname{lcm}(ac, bc) = c \cdot \operatorname{lcm}(a, b)$$

Theorem 2.5.5

If a number is divisible by two numbers a and b, it will also be divisible by $\operatorname{lcm}(a, b)$.

Theorem 2.5.6 (Euclidean Algorithm)
The Euclidean algorithm states that

$$\gcd(x, y) = \gcd(x - ky, y)$$

where $x > y$ and k is a positive integer.

2.6 Modular Arithmetic

Theorem 2.6.1
If $a = x \pmod{n}$ and $b \equiv y \pmod{n}$, then

$$ab \equiv xy \pmod{n}$$

Remark 2.6.2
This is useful when having to calculate the last few digits of a expression because we can then only have to consider that many digits for all of our terms.

Theorem 2.6.3
If $a \equiv x \pmod{n}$, then

$$a^m \equiv x^m \pmod{n}$$

Remark 2.6.4
This is true since a^m is just $a \times a \times \cdots \times a$ and x^m is just $x \times x \times \cdots \times x$, so it follows from the product rule above.

2.7 Quadratic Factorizations

Theorem 2.7.1 (Difference of Squares)

$$x^2 - y^2 = (x - y)(x + y)$$

Theorem 2.7.2 (Binomial Square Expansions)

$$(x+y)^2 = x^2 + 2xy + y^2 = (x-y)^2 + 4xy$$

$$(x-y)^2 = x^2 - 2xy + y^2 = (x+y)^2 - 4xy$$

$$(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$$

$$(x+y)^2 - (x-y)^2 = 4xy$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz)$$

2.8 Simon's Favorite Factoring Trick

Theorem 2.8.1 (Simon's Favorite Factoring Trick)

$$xy + kx + jy + jk = (x+j)(y+k)$$

Remark 2.8.2
You can generally apply this factorization when you have xy, x, and y terms. After applying the factorization, you can then find all possible values for each of your terms in your factorization (remember negatives!).

Theorem 2.8.3 (Difference of Cubes)
$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Theorem 2.8.4 (Sum of Cubes)
$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

Theorem 2.8.5 (Binomial Cube Expansions)

$$(x+y)^3 = x^3 + 3xy(x+y) + y^3$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3xy(x-y) - y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

2.9 Chicken McNugget Theorem

Theorem 2.9.1 (Chicken McNugget Theorem)
The maximum value that cannot be expressed as the sum of non-negative multiples of a and b is $ab - a - b$ if a and b are relatively prime.

Remark 2.9.2
This theorem is useful in finding solutions to problems like ”the maximum amount of money that can’t be created with 3 cent and 5 cent coins”.

Chapter 3

Combinatorics

3.1 Permutations

Theorem 3.1.1
The number of ways of arranging n distinct objects is just

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Theorem 3.1.2 (Permutations Formula)
The number of ways to order k objects out of n total objects is

$$^{n}P_k = \frac{n!}{(n-k)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-(k-1))$$

Theorem 3.1.3 (Circular Arrangements)
The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct is $(n-1)!$
The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct and reflections of the same arrangement aren't considered distinct is $\frac{(n-1)!}{2}$

Remark 3.1.4
The reason that this is true is because we can simply fix 1 person to be at the top and there are $(n-1)!$ ways to arrange the other people. This accounts for rotations

since rotating an arrangement will result in someone else on top. We divide by 2 for reflections because of symmetry on both the left and right sides of the person chosen to be at the top.

3.2 Combinations

Theorem 3.2.1 (Combinations Formula)
The number of ways to choose k objects out of a total of n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdot \dots \cdot (n-k+1)}{k!}$$

.

Theorem 3.2.2 (Binomial Identity)
The binomial identity states that

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

Remark 3.2.3
Both of the above methods can be used for calculating the number of ways of choosing any # of items from n different items.

3.3 Word Rearrangements

Theorem 3.3.1 (Word Rearrangements)
The number of ways to order a word is

$$\frac{n!}{d_1! \times d_2! \times d_3! \times \dots}$$

where n is the number of letters and $d_1, d_2, d_3, \dots$ are the number of times each of the duplicate letters that appear in the word.

Remark 3.3.2
This may seem complicated, but it essentially means the number of rearrangements of an n-letter word is n! However, since some letters may appear multiple times (let's say d times), we must divide by d! for all such letters because there are d! ways to arrange the duplicate letters that we are overcounting.

3.4 Probability

Theorem 3.4.1 (Probability)

$$\text{probability} = \frac{\text{Total number of desired outcomes}}{\text{Total number of possible outcomes}}$$

3.5 PIE

Theorem 3.5.1 (Principle of Inclusion Exclusion for 2 Sets)
Given two sets, $|A_1|$ and $|A_2|$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Basically, we count the number of possibilities in 2 ”things” and subtract the duplicates.

3.6 PIE for 3 Events

Theorem 3.6.1 (Principle of Inclusion Exclusion for 3 Sets)
Given three sets, $|A_1|$, $|A_2|$, $|A_3|$,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

In this formula, we count the number of possibilities in 3 ”things”, subtract the possibilities that are duplicates in all 3 pairs of sets, and add back the number of duplicates that are in all 3 sets.

Theorem 3.6.2 (Principle of Inclusion Exclusion Generalized Formula)
Stated more formally, if $(A_i)_{1\leq i\leq n}$ are finite sets, then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

.

3.7 Stars & Bars

Theorem 3.7.1 (Stars and Bars)
The number of ways to place n indistinguishable objects into k distinguishable bins is

$$\binom{n+k-1}{n}$$

Remark 3.7.2
The reason this is true is because you can consider placing $k-1$ bars in between n objects which would be equivalent to having $n+(k-1)$ slots and choosing $k-1$ slots from those. This can be written as:

$$\binom{n+k-1}{k-1}$$

which is equivalent to

$$\binom{n+k-1}{n}$$

This general idea is important to understand because tricky problems will often add additional conditions.

Remark 3.7.3
Make sure to remember that Stars and Bars only works if the objects that are being distributed are identical!

3.8 Geometric Counting

Theorem 3.8.1

The number of squares in a $n \times n$ grid of squares is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

.

Remark 3.8.2

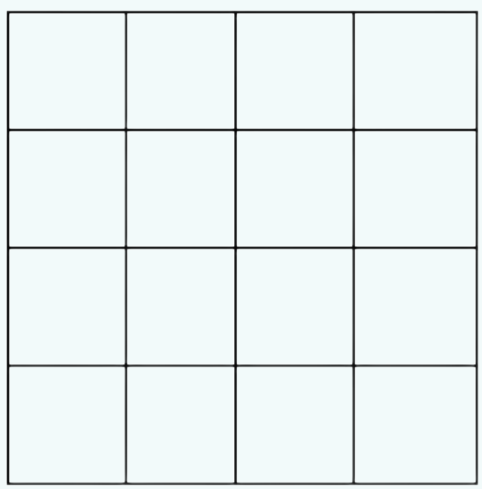
This comes from the fact that there are n^2 1x1 squares, $(n-1)^2$ 2x2 squares, ..., 2^2 (n - 1)x(n - 1) squares, and 1^2 nxn square.

Theorem 3.8.3

The general formula for the number of rectangles of all sizes in a rectangular grid of size $m \times n$ is

$$\binom{m+1}{2} \times \binom{n+1}{2}$$

.



Remark 3.8.4

Each combination of two horizontal lines and two vertical lines creates a unique rectangle.

We have

$$\binom{m+1}{2}$$

ways to choose two horizontal lines and

$$\binom{n+1}{2}$$

ways to choose two vertical lines.

Theorem 3.8.5

The number of ways to get from (0,0) to a point (m,n) moving only up and right is

$$\binom{m+n}{m}$$

Remark 3.8.6

Imagine any arrangement string of m R's and n U's. Each string would correspond to a unique path, so to count this, we can use the word rearrangement formula.

Chapter 4

Algebra

4.1 Work and Rate

Theorem 4.1.1

$$\text{Work} = \text{Rate} \times \text{Time}$$

Equivalently,

$$\text{Rate} = \frac{\text{Work}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Work}}{\text{Rate}}$$

Theorem 4.1.2

The if one person can do something is a amount of time, and someone else can do it in b amount of time, together they can do it in

$$\frac{ab}{a+b}$$

time.

Remark 4.1.3

This not only applies to work. For example, if a problem says 2 faucets take a and b

hours to fill a tub, together they can fill a tub in

$$\frac{ab}{a+b}$$

hours.

4.2 System of Equations

Theorem 4.2.1 (Quadratic Formula)
The solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Remark 4.2.2
If you are new to quadratics, I recommend checking out the following links:

Basics of factoring Quadratics

Basics of Quadratic Formula

Remark 4.2.3 (Solving 2 variable equations)
The most common ways of solving equations are substitution and elimination.
Linear Equations Basics

There is an easier way to solve 2 variable equations using the diagonal product method
Diagonal Product Method to Solve 2 Variable Equations

4.3 Speed, Distance, Time

Theorem 4.3.1

$$\text{Distance} = \text{Speed} \times \text{Time}$$

Equivalently,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

Theorem 4.3.2

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

Remark 4.3.3
A common mistake is to assume that average speed is the averages of all speeds (especially when the distance you are traveling at each of those speeds are the same). Remember, that's not true unless you are traveling at those speeds for the same amount of time!

Remark 4.3.4
Often, we may have to form equations with distance, speed, and time and use them to solve our problem.

4.4 Arithmetic Sequence

Theorem 4.4.1 (nth term in an Arithmetic Sequence)

$$a_n = a_1 + (n-1)d$$

which basically means the nth term of an arithmetic sequence is equal to

$$\text{first term} + (\text{number of terms} - 1)(\text{common difference})$$

We also have that

$$a_n = a_m + (n-m)d$$

which means

$$\text{the nth term} = \text{mth term} + (\text{number of terms} - m)(\text{common difference})$$

Theorem 4.4.2 (Number of Terms in an Arithmetic Sequence)

$$n = \frac{a_n - a_1}{d} + 1$$

Essentially,

$$\text{Number of Terms} = \frac{\text{Last Term} - \text{First Term}}{\text{Common Difference}} + 1$$

Theorem 4.4.3 (Average of Terms in an Arithmetic Sequence)

$$\text{Average of Terms} = \frac{a_1 + a_n}{2}$$

$$\text{Average of Terms} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

which essentially means

$$\text{Average of Terms} = \frac{\text{First Term} + \text{Last Term}}{2}$$

$$\text{Average of Terms} = \frac{\text{Sum of all Terms}}{\text{Number of Terms}}$$

$$\text{If the number of terms is even, } x = \text{average of middle 2 terms}$$

.

$$\text{If number of terms is odd, } x = \text{middle term}$$

Theorem 4.4.4 (Sum of all Terms in an Arithmetic Sequence)

$$S_n = \frac{a_1 + a_n}{2} \times n$$

which essentially means

$$\text{Sum of All Terms} = \text{Average of Terms } \times \text{Number of Terms}$$

We can also substitute

$$a_n = a_1 + (n-1)d$$

to get

$$S_n = \frac{2a_1 + (n-1)d}{2} \times n$$

Theorem 4.4.5 (Sum of Numbers Formula)

$$1 + 2 + 3 + \cdots + n = \frac{(n)(n+1)}{2}$$

Theorem 4.4.6 (Sum of Odd Numbers Formula)

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

In simple terms, the

$$\text{Sum of first n odd numbers} = n^2$$

Theorem 4.4.7 (Sum of Even Numbers Formula)

$$2 + 4 + 6 + \cdots + 2n = n(n+1)$$

To intuitively think about it, just take 2 common from each term

$$2(1 + 2 + 3 + \cdots + n) = 2\frac{(n)(n+1)}{2} = n(n+1)$$

In simple terms, the

$$\text{Sum of first n even numbers} = 2 \times \text{sum of first n numbers}$$

Theorem 4.4.8 (Sum of Squares Formula)

$$1^2 + 2^2 + \cdots + n^2 = \frac{(n)(n+1)(2n+1)}{6}$$

Theorem 4.4.9 (Sum of Cubes Formula)

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{(n)(n+1)}{2}\right)^2$$

Theorem 4.4.10 (nth term in a Geometric Sequence)

$$g_n = g_1 \cdot r^{n-1}$$

which basically means

$$\text{the nth term of a geometric sequence} = \text{first term} \times (\text{common ratio})^{\text{number of terms}-1}$$

A general form for calculating the nth term

$$g_n = g_m \cdot r^{(n-m)}$$

which basically means

$$\text{the nth term} = \text{mth term} \times \text{common ratio}^{(n-m)}$$

Theorem 4.4.11 (Number of Terms in a Finite Geometric Sequence)

$$n = \log_r \left(\frac{g_n}{g_1}\right) + 1$$

Essentially,

Number of Terms = 1 more than the number of times we needed to multiply$r$ by g_1 to get g_n

Theorem 4.4.12 (Sum of all Terms in a Finite Geometric Sequence)

$$S_n = g_1 \frac{(1 - r^n)}{1 - r}$$

which essentially means

$$\text{Sum of All Terms} = \text{First Term} \times \frac{1 - \text{common ratio}^n}{1 - \text{common ratio}}$$

Theorem 4.4.13 (Sum of all Terms in an Infinite Geometric Sequence)
For $-1 < r < 1$,

$$S_\infty = \frac{g_1}{1 - r}$$

Remark 4.4.14

The reason the formula only works for $|r| < 1$ is because if $|r| \geq 1$ the sum will diverge or essentially be infinite. We can only find the sum of a converging geometric sequence for which the sum approaches a constant value. Some Examples:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$$

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

4.5 Mean, Median, Mode Fundamentals

Definition 4.5.1 (Mean/Average)**.**

$$\text{Mean} = \text{average of all terms} = \frac{\text{sum of all terms}}{\text{number of terms}}$$

Remark 4.5.2

Often, in these types of problems, we can simply consider the difference between each value and the mean and make sure that value sums to 0.

Definition 4.5.3 (Mode)**.**

$$\text{Mode} = \text{Most common term(s)}$$

Remark 4.5.4

There could be multiple modes. If the problem says “unique mode”, it means that there is only one mode.

Definition 4.5.5 (Median)**.** After arranging the numbers in increasing or decreasing order: If number of terms is odd,

$$\text{Median} = \text{middle number}$$

If number of terms is even,

$$\text{Median} = \text{average of middle two numbers}$$

Definition 4.5.6 (Range)**.**

$$\text{Range} = \text{Largest number} - \text{Smallest Number}$$

4.6 Telescoping Basics

Concept 4.6.1 (Telescoping)

Expand the first few and last few terms, and cancel out any terms you see.

Remark 4.6.2

Generally, whenever you have long expressions that seem to be hard or impossible to compute manually, telescoping is probably at play.